DAVID W. TAYLOR NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER

Bethesda, Md. 20084

CONTROL THEORY WITH APPLICATIONS TO
NAVAL HYDRODYNAMICS
THE FIRST
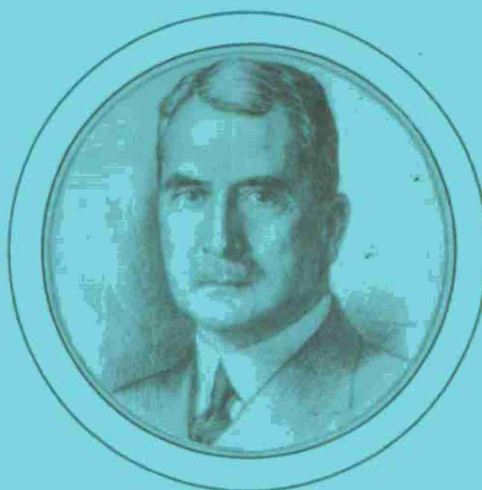DAVID W. TAYLOR LECTURES
APRIL 1972

by

Professor Dr. Reinier Timman

Technische Hogeschool
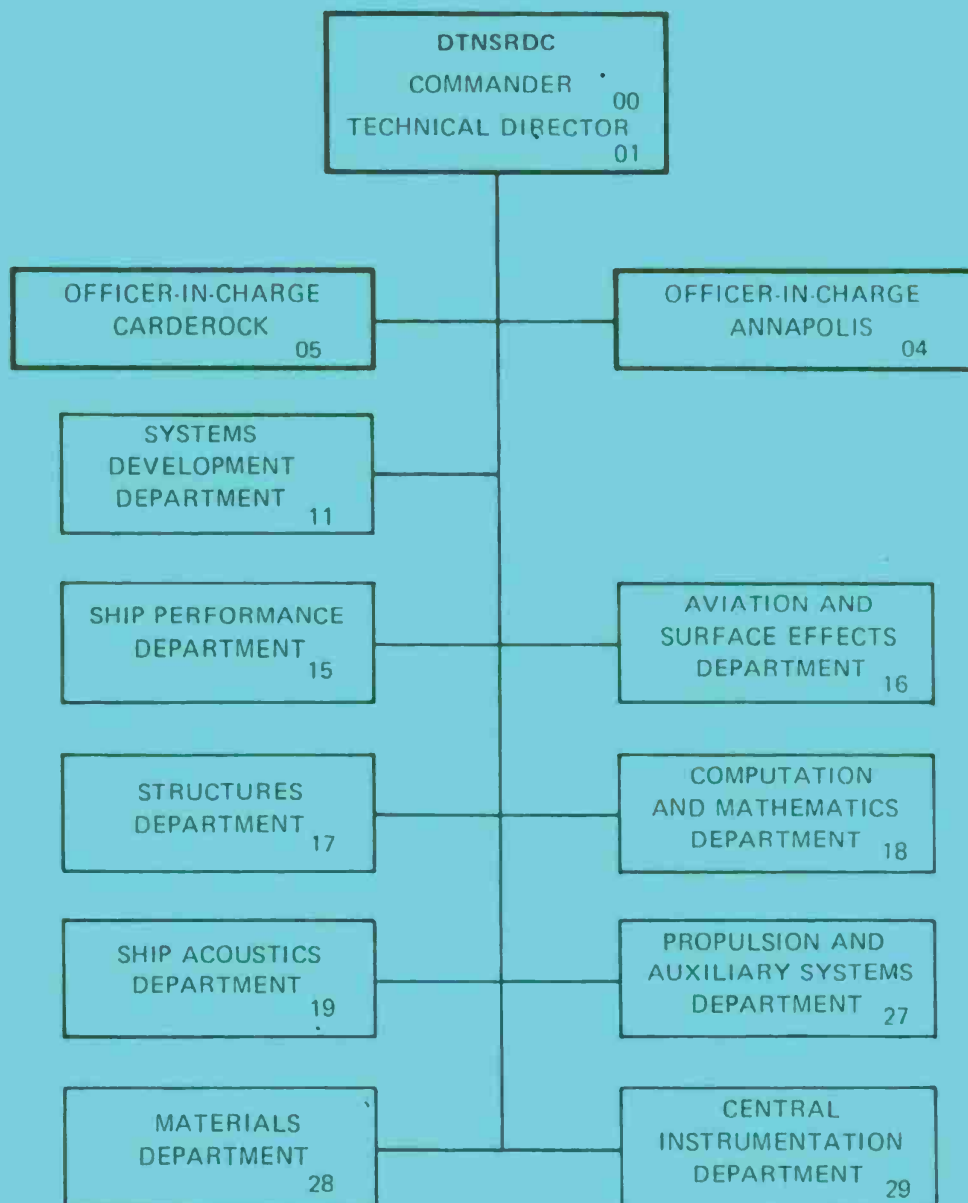Delft, Netherlands

Notes by
Thomas J. Langan

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Report 4397

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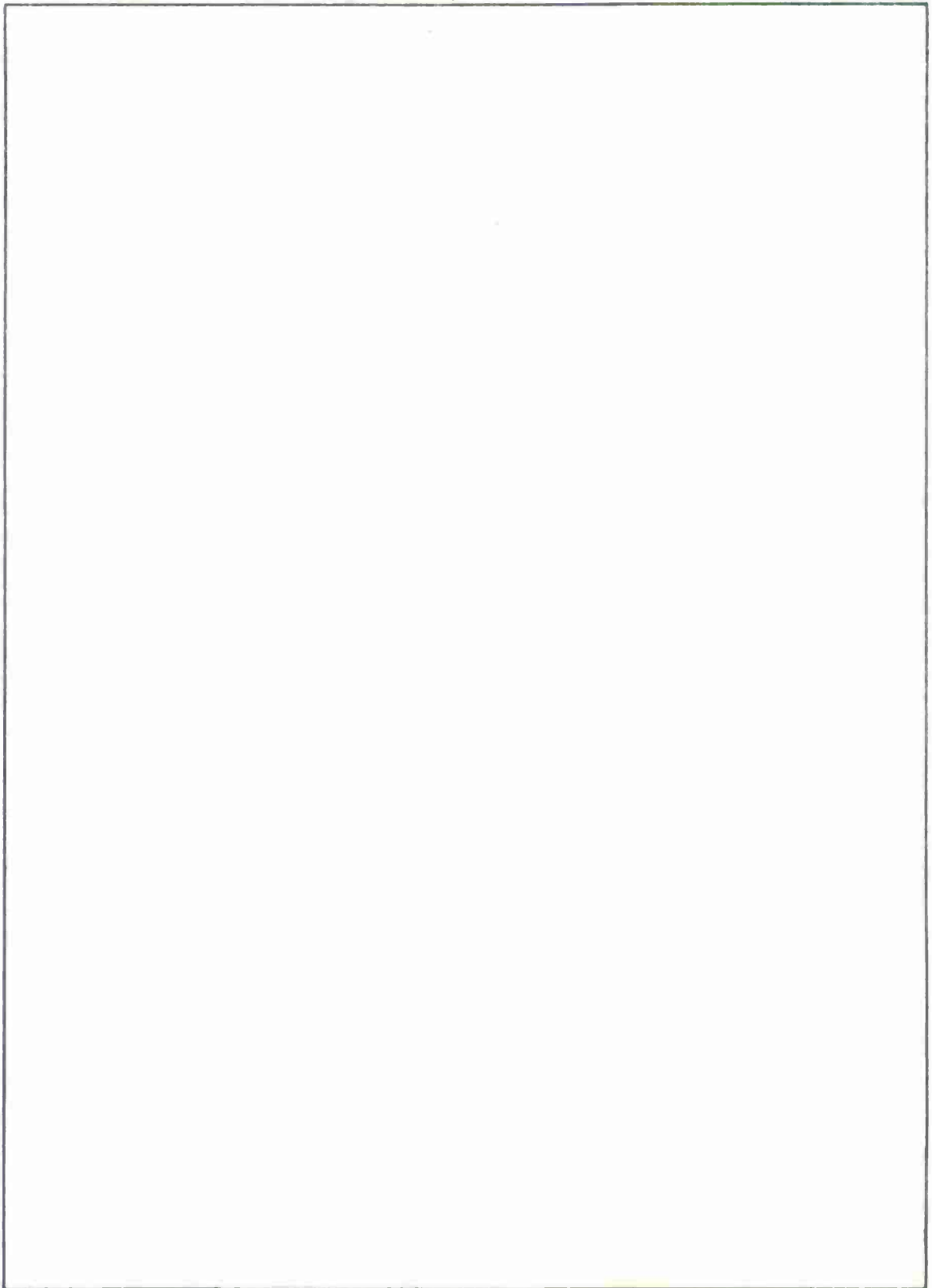| REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)* The lectures present an introduction to modern control theory. Calculus of variations is used to study the problem of determining the optimal control for a deterministic system without constraints and for one with constraints. The method of dynamic programming is also used to solve the unconstrained control problem. Stochastic systems are introduced, and the Kalman-Bucy filter is derived.		

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PREFACE

The David W. Taylor Lectures were initiated as a living memorial to
our founder, in recognition of his many contributions to the science of
naval architecture and naval hydromechanics. His systematic investiga-
tion of resistance of ship hulls is universally known and used, but of
equal importance was his use of hydrodynamic theory to solve practical
problems. Many of the experimental techniques which he pioneered are
still in use today (for example, the use of a spherical pitot tube for
exploring the structure of a wake field). The system of mathematical
lines developed by Taylor was used to develop many designs for the Navy
long before the computer was invented. And perhaps most important of
all, he established a tradition of applied scientific research at the
"Model Basin" which has been carefully nurtured through the decades, and
which we treasure and protect today.

These lectures were conceived to support and strengthen this
tradition. We will invite eminent scientists in fields closely related
to the Center's work to spend a few weeks with us, to consult with and
advise our working staff, and to give lectures on subjects of current
interest.

It is most fitting that Professor Reinier Timman, mathematician and
philosopher, initiate this series. He has long been a friend and on
several occasions has used the Center for a retreat, to his benefit and
ours. He has inspired and advised our staff and cooperated in our work.
His students at Delft have made leading contributions to the development
of modern naval hydrodynamics. Professor Timman's belief that mathe-
matics can contribute powerfully to our technology is much in the David
Taylor tradition. We are honored that he agreed to give the first in
this David W. Taylor Lecture Series.

W. E. CUMMINS

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FOREWORD

It is great honor to me to be invited to give the first
in the series of David W. Taylor Lectures. My associations
with the Model Basin date from a long time ago, and a visit
to the United States is for me not a real visit unless I
have the opportunity to taste once more the stimulating
atmosphere which not only gives the Model Basin an out
standing place in hydrodynamical research but also acts as
a breeding ground where nearly all outstanding people in the
field passed an essential period in their lives. So I am
extremely grateful to have been given the opportunity once
more to spend some time at this most interesting place and
to participate in its work. I wish to express my gratitude
to Justin McCarthy who originated the idea of the lectures
and to all other friends who made this period a success.
In particular, I am pleased that Dr. Langan, whom I used to
know as a promising undergraduate student, did a fine job
in editing the lectures.

R. TIMMAN
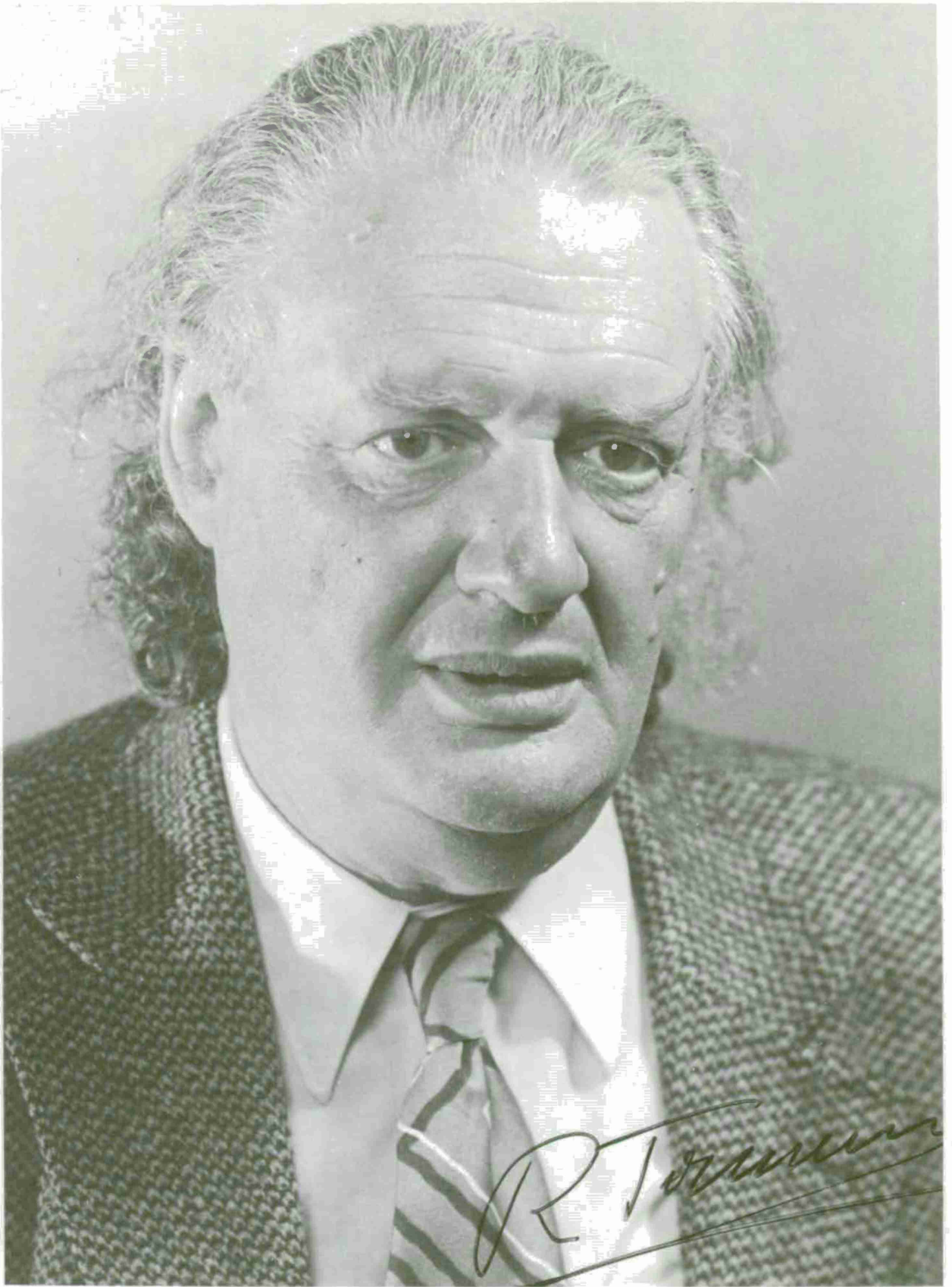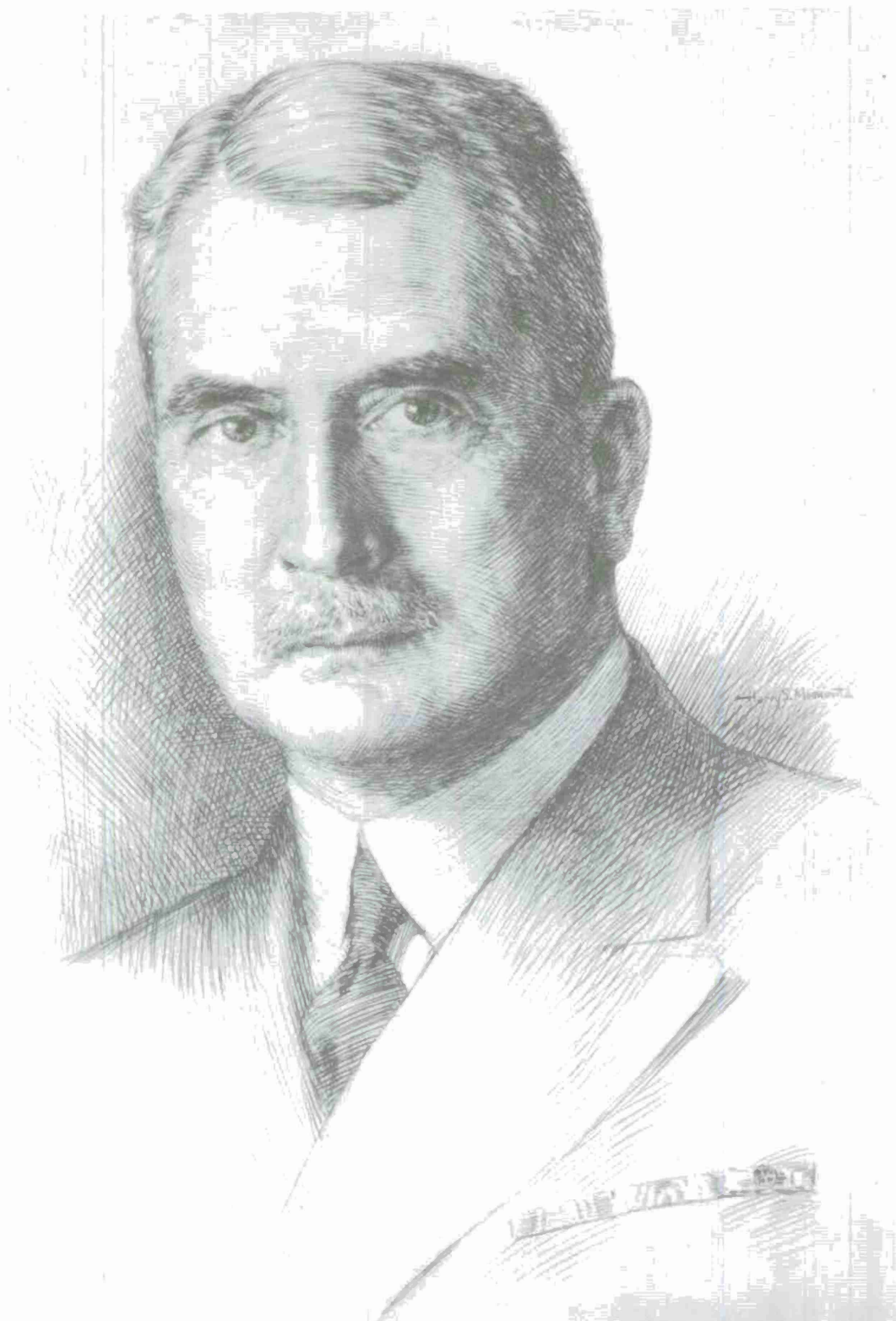
ABSTRACT

The lectures present an introduction to modern control
theory. Calculus of variations is used to study the problem
of determining the optimal control for a deterministic sys-
tem without constraints and for one with constraints. The
method of dynamic programming is also used to solve the
unconstrained control problem. Stochastic systems are intro-
duced, and the Kalman-Bucy filter is derived.

INTRODUCTION

Optimal control theory is involved with the great human effort to
control or influence processes of one type or another. The objectives
and criteria for the performance of a physical system may be diffused or
defy tractable analysis in many situations, but the basic concepts on
which to proceed have been established in control theory. One first
considers a system and a process through which the state of the system
is changing in time; in other words, some action or motion of the system
takes place in time. This behavior of the system is described by a set
of time-dependent variables $\underline{x}(t) = (x_1, \ldots, x_n)$ which are called
the state variables. In addition to the state of the system, one also
considers controls by which the process in question can be influenced.
These controls are represented by a set of variables $\underline{u}(t) =$
$(u_1(t), \ldots, u_m(t))$ which are called the control variables.

At a certain instant in time, say t_0, the state of the system is
known to be x_0. If an analysis of the system is to be performed, a sys-
tem of equations must be specified which predict the state for $t > t_0$
and for a given control function $\underline{u}$. These equations are called the
dynamic equations for the system; they may take the form of an ordinary
differential equation

$$\underline{\dot{x}}(t) = \underline{f}(t, \underline{x}, \underline{u})$$

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or a difference equation

$$x_{n+1} = f(t_n, x_n, u_n)$$

They might even take the form of an integro-differential-difference equation or a time delay equation, but they cannot take on a form such that, the solution at some time t_1 is dependent on the solution in the future, $t > t_1$. The dynamic equations must reflect this principle of nonanticipation. One does not violate this principle by choosing a control in anticipation of the future and thus influencing the future state of the system based on estimated future information; in fact, the choice of such a control is actually based on the history of the state of the system available at the time of the choice.

If no further specification of system performance is given, every control function which yielded a physical realizable state of the system for $t > t_0$ would be a solution to the control problem. One can have a meaningful control problem only if there is a desired objective, a goal to be achieved by the process. Moreover, it is not sufficient merely to have a goal; there must be a control by which this goal can be achieved. This control could be the case of no control, $f(t, x, \underline{u}) = f(t, x)$; however, it must exist. Since it is not the purpose of these notes to delve into all the mathematical problems, it will be assumed that there exists at least one control by which the objective can be achieved. It will further be assumed that any control function used in the sequel yields a unique state function $\underline{x}(t)$ with $\underline{x}(t_0) = x_0$; the state function is obtained by solving the dynamic equations.

In general, there are a number of controls which could yield the desired system state. From among this set of possible controls, one would like to choose the "best" control with respect to some performance criterion. For example, one would like to choose the control so that the process is carried out with a minimum cost in fuel, or time, or money. In the sequel, it is assumed that the performance criterion can be expressed in terms of a cost function; furthermore, it is assumed that the cost function is additive with respect to the contribution from each time interval. An example of such a cost function is

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$$G(\underline{x}_T, \ T) \ + \ \int_{t_0}^{T} \ F(\sigma, \ \underline{x}, \ \underline{u}) \ d\sigma$$

where $\underline{x}_T = \underline{x}(T)$. This cost function is dependent on the final state of
the system through the function G and on the intermediate states and the
control function through the function F. The additive property of the
control function with respect to the intermediate times is represented
by the integral. By an optimal control is meant that control which
minimizes the cost function; it is this function which is the desired
result of optimal control theory.

Any process that is being controlled is subject to unpredicted
disturbances, and these can make a significant difference in the choice
of a control function. Suppose the dynamic equations of a system is
given by the differential equation

$$\frac{dx}{dt} \ = \ u \ + \ p(t)$$

where $p(t)$ represents a disturbance. The behavior of the system in
response to the two different controls $(u_1 = - \ x)$ and $(u_2 = - \ e^{-t})$ does
not differ if there is no disturbance $(p \equiv 0)$; however, if a disturbance
is present, the response is significantly different. If $x_0 = 1$, the
response to the first control is given by

$$x_1 = e^{-t} + e^{-t} \ \int_{0}^{t} \ e^{\sigma} p(\sigma) \ d\sigma$$

whereas the response to the second control is

$$x_2 = e^{-t} + \int_{0}^{t} \ p(\sigma) \ d\sigma$$

Such differences could conceivably result in a different choice for an
optimal control.

In analyzing systems and their control, one must find a way to represent the unpredictable disturbances. Such disturbances cannot be modeled by analytic functions since the value of an analytic function at any point is predictable from its value on an arbitrary short interval. One answer to modeling these disturbances is to describe them as stochastic processes.[1] The theory of such processes was developed to model the fluctuation observed in physical systems. Wiener processes or the Brownian motion process are of particular interest to the stochastic control problem; many of the disturbances that affect a control system can be modeled by processes generated from Wiener processes. A Wiener process is a stochastic process in which the statistical properties over the interval $(t, t+\tau)$ are the same as those over the interval $(s, s+\tau)$; moreover, the behavior of the process is independent over time intervals which do not overlap, and there is no trend in the behavior.

Once the stochastic disturbances have been introduced into the control theory, the problem is no longer deterministic. The state variables and control variables are no longer predictable but must be described by their statistical properties. Kalman and Bucy[2] provide a solution to the stochastic control problem for nonstationary linear systems. Their solution consists of using an optimal filter to estimate from the observed system performance the state of the system in terms of the conditional mean; the estimated state is fed back to the control signal through linear feedback. The linear feedback is determined by solving a deterministic control problem; the filter depends on the disturbances and on the system dynamics, but it is independent of the cost. Although the nonlinear stochastic control problem or its equivalent, the nonlinear filter problem, has not been solved, some headway has been made by Bucy and Joseph;[3] this lecture considers only the linear problem.

[1] Astrom, K. J., "Introduction to Stochastic Control Theory," Academic Press, Inc., New York (1970).

[2] Kalman, R. E. and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," Journal of Basic Engineering Series D, American Society of Mechanical Engineers, Vol. 83, pp. 95--108 (1961).

[3] Bucy, R. S. and P. D. Joseph, "Filtering for Stochastic Processes with Applications to Guidance," Interscience Publishers, Inc., New York (1968).

As an example of a control problem, consider a ship moving through a current of water; the ship is a system undergoing a change in state. In this example, the state is the position (x, y) of the ship. The parameters which control the motion of the ship are the power, which determines the velocity relative to the water, and the steerage angle, which controls the heading angle θ. In this simplification of the system, the dynamic equations are:

$$\dot{x} = V \cos \theta + u(x, y)$$

$$\dot{y} = V \sin \theta + v(x, y)$$

where u and v are the velocity of the current in the x- and y-directions, respectively. The goal might be to go from point A to point B. If it is desired to reach B in the shortest possible time, the cost function would be the accumulated time; if it is desired to reach B with the minimum expenditure in fuel, the cost function would give the expended fuel in terms of x, y, V, and θ. A more complicated cost function would result if it is desired to reach B in the least time with a reasonable expenditure of fuel. Both the power and steerage angle could be subject to unpredictable perturbations; there could also be a stochastic perturbation of the current.

This lecture on control theory first treats a deterministic optimal control problem with no constraints on the controls. It is first solved by transforming the problem into a boundary-value problem for an ordinary differential equation, the so called indirect approach; it is then solved by the direct method developed by Bellman, the method of dynamic programming.[4] The big contribution of modern control theory to the deterministic control problem has been the extensions to controls with constraints, and a discussion of constrained controls constitutes another major topic of the lecture. Still another important area is the development of the theory of stochastic processes necessary in the

[4]Bellman, R. E. and S. E. Dreyfus, "Applied Dynamic Programming," Princeton University Press, Princeton, N.J. (1962).

treatment of stochastic controls. Finally, the theory of Kalman-Bucy filters is given and their solution to the stochastic control problem is presented for linear systems.

THE OPTIMAL CONTROL PROBLEM

In these lectures the simplest optimal control problem considered is that of a state variable $x(t)$ and a control variable $u(t)$ defined on an interval $0 \leq t \leq T$. The process being controlled is described by the dynamic equations

$$\underline{\dot{x}}(t) = \underline{f}(t, \underline{x}, \underline{u}) \tag{1.1}$$

with

$$\underline{x}(0) = x_0 \tag{1.2}$$

The vector $\underline{f}$ is twice continuously differentiable with respect to $\underline{x}$ and Lipschitz continuous with respect to $\underline{u}$; this latter condition means simply that there is a constant L such that for every pair of control vectors $\underline{u}$ and $\underline{v}$

$$\left| \underline{f}(t, \underline{x}, \underline{u}) - \underline{f}(t, x, \underline{v}) \right| \leq L \left| \underline{u} - \underline{v} \right| \tag{1.3}$$

For each control vector $\underline{u}$, these conditions imply that the state vector $\underline{x}$, which is obtained from solving (1.1) and which also satisfies the initial condition (1.2), exists and is unique. Moreover, from among the set of control vectors, it is assumed that there is a unique control $\underline{u}$ which minimizes the cost function C_T. The cost function is defined by the following:

$$C_T[\underline{u}] = G(\underline{x}_T, T) + \int_0^T F(\sigma, \underline{x}, \underline{u}) \, d\sigma \tag{1.4}$$

The functions F is twice continuously differentiable with respect to $\underline{x}$ and Lipschitz continuous with respect to $\underline{u}$; G represents the cost at the terminal point $\underline{x}(T) = x_T$; it is twice continuously differentiable with respect to X_T.

Suppose that v is an optimal control vector, and consider a slight deviation $\delta\underline{u}$ of this control vector. If

$$\underline{u}(t) = \underline{v} + \delta\underline{u}$$

$\underline{u}(t)$ is also a control vector, as can be seen from an application of the theory of ordinary differential equations. If $\underline{z}$ is the state vector associated with the control $\underline{v}$, the new control $\underline{u}$ yields a new state vector $\underline{x}$ given by

$$\underline{x}(t) = \underline{z} + \delta\underline{x}$$

where $\delta\underline{x}$ is an unknown. Moreover, since $\underline{v}$ minimizes the cost function, the new cost function is greater;

$$\int_0^T F(\sigma, \underline{x}, \underline{u}) \, d\sigma + G(\underline{x}_T, T)$$

$$\geq \int_0^T F(\sigma, \underline{z}, \underline{v}) \, d\sigma + G(\underline{z}_T, T) \tag{1.5}$$

Since the old state vector satisfies

$$\dot{\underline{Z}} = f(t, \underline{z}, \underline{v})$$

and the new one satisfies

$$\dot{\underline{x}} = \underline{f}(t, \underline{x}, \underline{u})$$

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then

$$\dot{\underline{x}} + \delta\dot{\underline{x}} = \dot{\underline{x}} = \underline{f}(t, \underline{z} + \delta\underline{x}, \underline{v} + \delta\underline{u})$$

Now by assumption $\underline{f}$ is twice continuously differentiable with respect to $\underline{x}$; hence

$$\delta\dot{\underline{x}} = \underline{f}(t, \underline{z} + \delta\underline{x}, \underline{v} + \delta\underline{u}) - \underline{f}(t, \underline{z}, \underline{v})$$

$$= \underline{f}_x \delta x + \underline{f}(t, z, \underline{v} + \delta\underline{u}) - \underline{f}(t, \underline{z}, \underline{v}) + O(|\delta x|^2) \quad (1.6)$$

It is not necessary that $\delta\underline{u}$ be uniformly small; indeed, in problems involving bang-bang controls, this is not at all true. However, there can be deviations $\delta\underline{u}$ of order one only if their duration is short. It can be proved that if $\delta\underline{u}$ satisfies the condition

$$\int_0^T |\delta\underline{u}(\sigma)| \, d\sigma < \varepsilon \quad (1.7)$$

then the deviations $\delta\underline{x}(t)$ are also of order ε. Since by assumption, $\underline{f}$ is Lipschitz continuous with respect to $\underline{u}$,

$$\left|\underline{f}(t, \underline{z}, \underline{u}) - f(t, \underline{z}, \underline{v})\right| \le L|\underline{u} - \underline{v}| = O(\delta u)$$

Moreover, it follows from Equation (1.6) that to the same order of approximation

$$\delta\dot{\underline{x}} = \underline{f}_x \delta\underline{x} + \underline{f}(t, \underline{z}, \underline{u}) - f(t, \underline{z}, \underline{v}) \quad (1.8)$$

or in abbreviated form

$$\delta\dot{\underline{x}} = \underline{f}_x \delta\underline{x} + f(\underline{u}) - f(\underline{v}) \quad (1.9)$$

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This equation is a linear differential equation for $\delta\underline{x}$, and there are standard ways for solving linear differential equations.[5] One first considers the linear homogeneous equation

$$\dot{\underline{y}} = A\underline{y} \tag{1.10}$$

where in our case $\underline{y}$ represents the vector $\delta\underline{x}$ and A the matrix $\underline{f}_{\underline{x}}$. Let

$$\underline{y}^j(t) = (\Phi_{1j}(t, \tau), \Phi_{2j}(t, \tau), \ldots, \Phi_{nj}(t, \tau))$$

be the solution of Equation (1.10) with $\Phi_{ij}(\tau, \tau) = \delta_{ij}$, the Kronecker delta; moreover, let $\Phi(t, \tau)$ be the matrix whose column vectors are the vectors $\underline{y}^j$, $\Phi(t, \tau) = \Phi_{ij}(t, \tau)$. The matrix $\Phi(t, \tau)$ is called the transport matrix or fundamental matrix for the differential Equation (1.10). From (1.10) it follows that as a function of t

$$\frac{\partial\Phi}{\partial t}(t, \tau) = A\Phi(t, \tau) \tag{1.11}$$

and by its definition

$$\Phi(\tau, \tau) = I \tag{1.12}$$

where I is the unit matrix. The solution $\underline{y}(t)$ is given in terms of its value at $t = \tau$ by

$$\underline{y}(t) = \Phi(t, \tau)\,\underline{y}(\tau) \tag{1.13}$$

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[5] Coddington, E. A. and N. Levinson, "Theory of Ordinary Differential Equations," McGraw-Hill Book Company, Inc., New York (1955).

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Hence

$$I \underline{y}(t) = \underline{y}(t) = \Phi(t, \tau) \, y(\tau)$$

$$= \Phi(t, \tau) \, \Phi(\tau, t) \, y(t)$$

or if $\underline{y} \neq 0$,

$$I = \Phi(t, \tau) \, \Phi(\tau, t) \tag{1.14}$$

Differentiating with respect to t yields

$$0 = \frac{\partial \Phi}{\partial t} (t, \tau) \, \Phi(\tau, t) + \Phi(t, \tau) \frac{\partial}{\partial t} \, \Phi(\tau, t)$$

$$= A \, \Phi(t, \tau) \, \Phi(\tau, t) + \Phi(t, \tau) \frac{\partial}{\partial t} \, \Phi(\tau, t)$$

$$= A(t) + \Phi(t, \tau) \frac{\partial}{\partial t} \, \Phi(\tau, t)$$

It can be shown that $\Phi(t, \tau)$ has an inverse and that this inverse is $\Phi(\tau, t)$; consequently

$$\frac{\partial}{\partial t} \, \Phi(\tau, t) = - \, \Phi^{-1}(t, \tau) \, A(t) = - \, \Phi(\tau, t) \, A(t)$$

that is, $\Phi(t, \tau)$ as a function of τ satisfies

$$\frac{\partial}{\partial \tau} \, \Phi(t, \tau) = - \, \Phi(t, \tau) \, A(\tau) \tag{1.15}$$

Although (1.15) will be used subsequently, of immediate interest is the solution to the inhomogeneous linear equation

$$\dot{\underline{y}} = A \, \underline{y} + \underline{g}(t) \tag{1.16}$$

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with $\underline{y}(\tau) = 0$; the solution is given by

$$\underline{y}(t) = \int_\tau^t \Phi(t, \sigma) \, g(\sigma) \, d\sigma \qquad (1.17)$$

which can be verified by substitution into (1.16). For the control problem, (1.17) has two consequences: it can be used in conjunction with (1.6) to obtain an estimate for the order of magnitude of $\delta\underline{x}$ and it can be used to solve (1.9).

In the first case,

$$|\delta\underline{x}| \leq \int_0^t |\Phi(t, \sigma)| \, | \, f(u) - f(\underline{v})| \, d\sigma + \int_0^t 0(\delta x^2) \, d\sigma$$

$$\leq M \int_0^t | \, f(\underline{u}) - f(\underline{v})| \, d\sigma + \int_0^t 0(\delta x^2) \, d\sigma$$

where M is a bound for Φ. From (1.3)

$$|\delta\underline{x}| \leq LM \int_0^t |\delta\underline{u}| \, d\sigma + \int_0^t 0(\delta x^2) \, d\sigma$$

$$\leq LM\epsilon + \int_0^t 0(\delta x^2) \, d\sigma$$

By iteration

$$|\delta\underline{x}| \leq LM\epsilon + \int_0^t 0(\epsilon^2) \, d\sigma = 0(\epsilon)$$

The second case is of more interest, of course, for it gives an approximation of $\delta\underline{x}$ good to the second order in ϵ, namely,

$$\delta\underline{x} = \int_0^t \Phi(t, \sigma) \, [\underline{f}(\underline{u}) - \underline{f}(v)] \, d\sigma \qquad (1.18)$$

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where Φ is defined by

$$\frac{\partial \Phi}{\partial t} (t, \tau) = \underline{f}_x(t) \ \Phi(t, \tau) \qquad (1.19)$$

Now consider the difference in the values of the cost function; by (1.5)

$$\int_0^t F(\sigma, \underline{x}, \underline{u}) - F(\sigma, \underline{z}, \underline{v}) \ d\sigma + G(x_T, T) - G(z_T, T) \geq 0$$

Hence, from the assumptions on F and G,

$$\int_0^T [F_x \delta\underline{x} + F(\underline{u}) - F(\underline{v})] \ d\sigma + G_x \delta\underline{x}(T) \geq 0$$

By (1.18),

$$\int_0^T [F_x(\tau) \int_0^\tau \Phi(\tau, \sigma) \ (\underline{f}(\underline{u}(\sigma)) - f(v(\sigma)))d\sigma + F(\underline{u}(\tau)) - F(\underline{v}(\tau))] \ d\tau$$

$$+ G_x \int_0^T \Phi(T, \sigma) \ (\underline{f}(\underline{u}(\sigma)) - \underline{f}(\underline{v}(\sigma))) \ d\sigma \geq 0 \qquad (1.20)$$

If the order of integration in the double integral is changed,

$$\int_0^T F_x(\tau) \int_0^\tau \Phi(\tau, \sigma) \ [f(\underline{u}(\sigma)) - f(\underline{v}(\sigma)) \ d\sigma d\tau$$

$$= \int_0^T \int_\sigma^T F_x(\tau)\Phi(\tau, \sigma) \ d\tau \ [f(\underline{u}(\sigma)) - f(\underline{v}(\sigma))] \ d\sigma \qquad (1.21)$$

The vector function $\underline{p}^T$ is defined by

$$\underline{p}^T(t) = - \int_t^T F_x(\tau) \ \Phi(\tau, t) \ d\tau - G_x(T) \ \Phi(T, t) \qquad (1.22)$$

Recall that one of the properties of Φ was (1.15)

$$\frac{\partial \Phi}{\partial t} (\tau, t) = - \Phi(\tau, t) \ \underline{f}_x(t)$$

Then

$$\dot{\underline{p}}^T = F_x(t) \ \Phi(t, \ t) \ - \ \int_t^T F_x(\tau) \ \frac{\partial}{\partial t} \ \Phi(\tau, \ t) \ d\tau \ - \ G_x(T) \ \frac{\partial}{\partial t} \ \Phi(T, \ t)$$

$$= F_x(t) \ + \ \int_0^T F_x(\tau) \ \Phi(\tau, \ t) \ f_x(t) \ d\tau \ + \ G_x(T) \Phi(T, \ t) \ f_x(t)$$

$$= F_x(t) \ - \ - \left[\ \int_t^T F_x(\tau) \ \Phi(\tau, \ t) \ d\tau \ - \ G_x(T) \ \Phi(T, \ t) \right] f_x(t)$$

$$\dot{\underline{p}}^T = F_x - \underline{p}^T \underline{f}_x \tag{1.23}$$

with $\underline{p}^T(T) = G_x(T)$. In terms of p^T, (1.20) becomes

$$\int_0^T [- p^T(\sigma) \ (\underline{f}(\underline{u}) - \underline{f}(\underline{v})) + F(\underline{u}) - F(\underline{v})] \ d\sigma \geq 0$$

or

$$\int_0^T [- F(\underline{v}) + \underline{p}^T \underline{f}(\underline{v})] - [- F(\underline{u}) + p^T \underline{f}(\underline{u})] \ d\sigma \geq 0 \tag{1.24}$$

Since δu is an arbitrary deviation satisfying only (1.7), it can be chosen such that $\underline{u} = \underline{v}$ everywhere except on some arbitrary interval; as a consequence, the inequality in (1.24) must hold for the integrand:

$$- F(\underline{v}) + p^T \underline{f}(\underline{v}) \geq - F(\underline{u}) + p^T \underline{f}(\underline{u})$$

Define

$$H(t, \ \underline{u}) = - F(\underline{u}) + \underline{p}^T \underline{f}(\underline{u}) \tag{1.25}$$

Then H satisfies

$$H(t, \ \underline{v}) \geq H(t, \ \underline{u}) \tag{1.26}$$

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for v, an optimal control. This is the Pontryagin maximal principle which states that for given values of p^T and $\underline{x}$ at time t, the optimal control $\underline{v}(t)$ is the control function for which the Hamiltonian $H(t, u)$ is a maximum.

If the control functions are sufficiently smooth, the optimal control is that control for which

$$0 = \frac{\partial H}{\partial u} = - F_u + p^T f_u \tag{1.27}$$

It is assumed that f is differentiable with respect to $\underline{u}$; prior to this equation, f need only be Lipschitz continuous with respect to u. This equation is a system of m equations which could be solved for the m control functions $(u_1, \ldots, u_m)$ in terms of the state variables $(x_1, \ldots, x_n)$ and the new variables $(p_1^T, \ldots, p_n^T)$. Consequently, the optimal control problem has been reduced to a two-point, boundary-value problem for an ordinary differential equation:

$$\underline{\dot{x}} = f(t, \underline{x}, \underline{u})$$

$$\dot{p}^T = F_x - p^T f_x$$

$$0 = - F_u + p^T f_u$$

or

$$\dot{x} = \frac{\partial H}{\partial p^T}$$

$$\dot{p} = - \frac{\partial H}{\partial x} \tag{1.28}$$

$$0 = \frac{\partial H}{\partial u}$$

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where

$$x(0) = x_0$$

$$p(T) = G_x(T) \qquad\qquad (1.29)$$

There are just enough conditions to determine $\underline{x}$, $\underline{p}^T$, and u.

The function H contains the variables $\underline{x}$, $\underline{p}^T$, $\underline{u}$, and of course t. Using (1.26) to eliminate u, (1.28) can be expressed in terms of the set of dual variables x and $\underline{p}' = \underline{p}^T$, where the prime denotes transpose of the vector; the resulting system is the familiar canonical form of classical mechanics.

$$\underline{\dot{x}} = \frac{\partial H}{\partial p}$$

$$\underline{\dot{p}} = -\frac{\partial H}{\partial x} \qquad\qquad (1.30)$$

The boundary conditions are stated in terms of $\underline{x}_0$, $\underline{x}_T$, and T; for instance, both $\underline{x}_T$ and T might be fixed, or either one might vary while the other is fixed. No boundary conditions are specified directly in terms of $\underline{p}$; the boundary conditions on $\underline{p}$ are obtained indirectly by substitution into (1.29). Equation (1.29) does, however, contain a sufficient set of conditions to pose a two-point, boundary-value problem for (1.30).

Another form that the boundary condition at t = T might assume is for $\underline{x}_T$ and T to satisfy an end condition of the form

$$\underline{M}(\underline{x}_T, \ T) = 0 \qquad\qquad (1.31)$$

where $\underline{M}$ is a twice continuously differentiable vector function of both its arguments. In this case the method of Lagrange multipliers will be used to transform the optimal control problem into a corresponding

two-point, boundary-value problem. The vector $\underline{q}$ is introduced here as a Lagrange variable. Now the problem of minimizing the cost function (14) is replaced by the problem of finding the unconstrained minimum of

$$C_q(\underline{u}) = \int_0^T F(\sigma, \underline{x}, \underline{u}) + \underline{q}'(\dot{\underline{x}} - \underline{f}(\sigma, \underline{x}, \underline{u}) \, d\sigma$$

$$+ \underline{\mu}'\underline{M}(\underline{x}_T, T) + G(\underline{x}_T, T) \quad (1.32)$$

The boundary condition (1.31) has been inserted into the cost function by means of the Lagrange multiplier $\underline{\mu}$. Suppose $\underline{v}(t)$ is the control which minimizes C_q. For a variation $\delta\underline{u}$ to the control $\underline{v}$, let $\underline{x}(t)$ denote the new state variable, and let $t = T+\Delta T$ be the time at the new terminal point. The main difference from the previous argument in this section is that the terminal time is $T+\Delta T$ rather than ΔT. The new cost is given by

$$C_q(\underline{u}) = \int_0^{T+\Delta T} F(\sigma, \underline{z} + \delta\underline{x}, \underline{v} + \delta\underline{u}) + \underline{q}'(\dot{\underline{z}} + \delta\dot{\underline{x}} - f(\sigma, \underline{z} + \delta\underline{x}, \underline{v} + \delta\underline{u}) \, d\sigma$$

$$+ G(\underline{x}(T + \Delta T), T + \Delta T) + \underline{\mu}'\underline{M}(\underline{x}(T + \Delta T), T + \Delta T)$$

Hence the increase in cost $C_q(\underline{u}) - C_q(\underline{v})$ is given as:

$$C_q(\underline{u}) - C_q(\underline{v}) = \int_0^T [F_x\delta\underline{x} + F_u\delta\underline{u} + \underline{q}'\delta\dot{\underline{x}} - \underline{q}'f_x\delta\underline{x} - \underline{q}'f_u\delta\underline{u}] \, d\sigma$$

$$+ G_x\Delta\underline{x}_T + G_T\Delta T + \underline{\mu}'(\underline{M}_x\Delta\underline{x}_T + \underline{M}_T\Delta T)$$

$$+ \int_T^{T+\Delta T} F(\sigma, \underline{x}, \underline{u}) + \underline{q}'\dot{\underline{x}} - \dot{\underline{q}}'\underline{f}(\sigma, \underline{x}, \underline{u}) \, d\sigma$$

where $\underline{x} = \underline{z} + \delta\underline{x}$

$$\int_0^T \underline{q}' \frac{d\delta\underline{x}}{dt} \, dt = \underline{q}'(T) \, \delta\underline{x}(T) - \int_0^T \dot{\underline{q}}' \, \delta\underline{x} \, d\sigma$$

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Hence

$$0 \leq \int_0^T \{[F_x - \underline{\dot{q}}' - \underline{q}'f_x] \, \delta\underline{x} + [F_u - \underline{q}'f_u] \, \delta\underline{u}\} \, d\sigma$$

$$+ \, \underline{q}'(T)\delta\underline{x}(T) + G_x\Delta\underline{x}_T + G_T\Delta T + \underline{\mu}'(\underline{M}_x\Delta\underline{x}_T + \underline{M}_T\Delta T)$$

$$+ \int_T^{T+\Delta T} F(\sigma, \, \underline{x}, \, \underline{u}) + \underline{q}'\underline{\dot{x}} - \underline{q}'f(\sigma, \, \underline{x}, \, \underline{u}) - F(T, \, \underline{x}_T \, \underline{u}_T)$$

$$- \, \underline{q}'\underline{\dot{x}}(T) + \underline{q}'\underline{f}(T, \, \underline{x}_T, \, \underline{u}_T) \, d\sigma$$

$$+ \, [F(T, \, \underline{x}_T, \, \underline{u}_T) + \underline{q}'\underline{z}_T + \underline{q}'\delta\underline{\dot{x}}(T) - \underline{q}'\underline{f}(T, \, \underline{x}_T \, \underline{u}_T)] \, \Delta T \qquad (1.33)$$

The integral from T to T+ΔT is a second order contribution which goes to zero faster than the other terms as $\Delta T \to 0$.

In order to determine $\Delta\underline{x}_T$, consider the solutions of the differential Equation (1.1), which have the initial value $\underline{x}_0$. These solutions satisfy the integral equation

$$\underline{x}(t) = \underline{x}_0 + \int_0^t \underline{f}(\sigma, \, \underline{x}, \, \underline{u}) \, d\sigma$$

Then

$$\Delta x_T = (\underline{x}(T) - \underline{z}(T)) + \int_T^{\Delta T} f(\sigma, \, \underline{x}, \, \underline{u}) \, d\sigma$$

$$= \delta\underline{x}(T) + \underline{f} \, \Delta T + 0(\varepsilon^2)$$

where the geometry of the proof is illustrated in Figure 1.
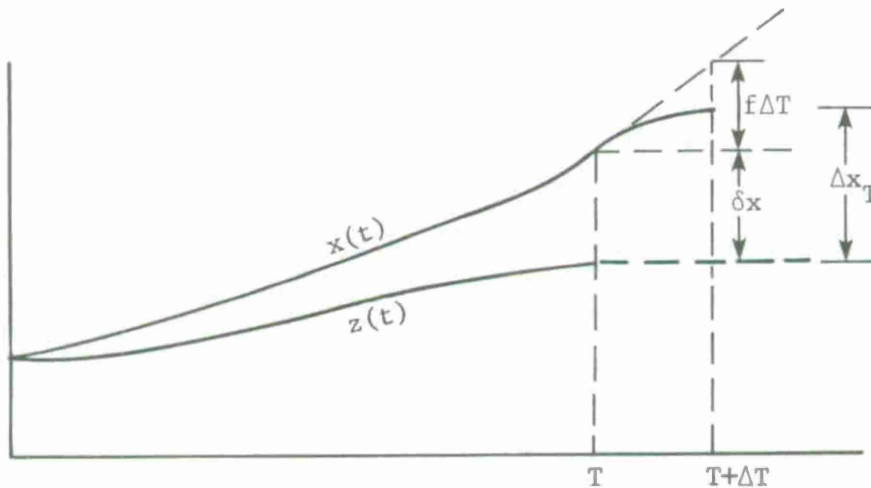
Figure 1 -- Geometry of the Proof

To within second order

$$\delta \underline{x}(T) = \Delta \underline{x}_T - \underline{f} \, \Delta T \qquad (1.34)$$

Within this order of approximation, (1.33) reduces to the following:

$$0 \le \int_0^T [F_x - \underline{\dot{q}}' - \underline{q}'f_x] \, \delta \underline{x} + [F_u - \underline{q}'\underline{f}_u] \, \delta \underline{u} \, d\sigma$$

$$+ [\underline{q}'(T) + G_x + \mu'\underline{M}_x] \, \Delta \, \underline{x}_T$$

$$+ [-\underline{q}'(T) \, \underline{f} + G_T + \mu'\underline{M}_T + F(T, \underline{x}_T, \underline{u}_T)] \, \Delta T$$

If $\underline{q}$ is now determined so that the coefficient of $\delta \underline{x}$ vanishes,

$$\underline{\dot{q}}' = F_x - \underline{q}'f_x$$

This is the same differential Equation (1.23) that $\underline{p}$ satisfied; our Lagrange multiplier can then be identified with $\underline{p}$

$$\underline{q} = \underline{p} \qquad (1.35)$$

Moreover, since the relationship must hold independent of $\delta \underline{u}$,

$$F_u - \underline{p}'\underline{f}_u = 0 \qquad (1.36)$$

Since there are no longer restrictions on $\Delta \underline{x}$ and ΔT,

$$\underline{p}'(T) + \underline{\mu}'\underline{M}_x + G_x = 0$$

$$(1.37)$$

$$F + G_T + \underline{\mu}'\underline{M}_T - \underline{p}'\underline{f} = 0$$

Introducing the Hamiltonian (1.25) yields

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$$\dot{\underline{x}} = \frac{\partial H}{\partial p}$$

$$\dot{\underline{p}} = -\frac{\partial H}{\partial x} \tag{1.28}$$

$$0 = \frac{\partial H}{\partial u}$$

The initial condition $\underline{x}(0) = \underline{x}_0$ together with the terminal conditions

$$\underline{M}(\underline{x}_T, T) = 0$$

$$\underline{p}'(T) = -(\underline{\mu}'\underline{M}_x + G_x) \tag{1.38}$$

$$H(T, \underline{u}(T)) = G_T + \underline{\mu}'\underline{M}_T$$

provides a sufficient number of conditions to determine $\underline{x}$, $\underline{p}$, $\underline{u}$, and T.
The last two equations in the system (1.38) are obtained from (1.37).

The problems of optimal control theory generally reduce to a two-
point, boundary-value problem for the system of ordinary differential
equations (1.30). Bailey, Shampine, and Waltman[6] discuss methods for
solving such two-point, boundary-value problems. These problems are
presently solved either by the shooting method or by solving a sequence
of simpler boundary value problems whose solutions converge to a solu-
tion of the given problem. In any case, very few of these problems can
be solved without the use of electronic computers either digital or
hybrid.

The shooting method is the easier, when it works. It consists of
supplementing the conditions at one end with a sufficient number of
assumed conditions to yield an initial value problem. The initial value

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[6] Bailey, P. B., et al., "Nonlinear Two-Point, Boundary-Value Problems,"
Academic Press, Inc., New York (1968).

problem is solved; the solution is substituted into the boundary conditions at the other end. If these conditions are satisfied, the solution to the initial value problem is the desired solution to the two-point, boundary-value problem; otherwise, a new set of assumptions is made based on the discrepancy between the actual boundary values and the calculated values. Hopefully, as one continues this iteration process, the solutions to the initial value problem converge to a solution of the two-point, boundary-value problem. The shooting method may not converge, or it can be unstable, that is, a small variation in the initial conditions results in a large variation in the solution. If the initial problem is unstable, a small error, such as roundoff on a computer, could cause subsequently computed values at another point to be meaningless.

Before proceeding to the direct method for solving the optimal control problem, take a second look at the Hamiltonian H and the functions $\underline{p}^T$. Suppose that the terminal cost G is identically zero; the cost function is then

$$C(\underline{u}) = \int_0^T F(\sigma, \underline{x}, \underline{u}) \, d\sigma$$

Further, assume that every point in an open neighborhood N of an optimal trajectory $\underline{z}(t)$ can be joined to the initial point $(0, \underline{x}_0)$ by a trajectory $\underline{x}(t)$ resulting from an optimal control. This assumption makes the minimal cost J a function of the terminal point $(T, \underline{x}_T)$ in N.

$$J(\underline{x}_T, T) = \text{Min} \int_0^T F(\sigma, \underline{x}, \underline{u}) \, d\sigma \qquad (1.39)$$

It is assumed that J is twice continuously differentiable. Then,

$$J(\underline{x}_T + \Delta\underline{x}_T, T+\Delta T) = J(\underline{x}_T, T) + J_x \Delta\underline{x}_T + J_T \Delta T \qquad (1.40)$$

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By the definition of J, there is a control $\underline{u} + \delta\underline{u}$ together with a trajectory $\underline{x} + \delta\underline{x}$ such that

$$J(\underline{x}_T + \Delta\underline{x}_T, \; T+\Delta T) = \int_0^T F(\sigma, \; \underline{x} + \delta\underline{x}, \; \underline{u} + \delta\underline{u}) \; d\sigma \tag{1.41}$$

where $\underline{u}$ is the control such that

$$J(\underline{x}_T, \; T) = \int_0^T F(\sigma, \; \underline{x}, \; \underline{u}) \; d\sigma \tag{1.42}$$

From (1.41) and (1.42)

$$J(\underline{x}_T + \Delta\underline{x}_T, \; T+\Delta T) - J(\underline{x}_T, \; T)$$

$$= \int_0^T F(\sigma, \; \underline{x} + \delta\underline{x}, \; \underline{u} + \delta\underline{u}) - F(\sigma, \; \underline{x}, \; \underline{u}) \; d\sigma$$

$$+ \int_T^{T+\Delta T} F(\sigma, \; \underline{x} + \delta\underline{x}, \; \underline{u} + \delta\underline{u}) \; d\sigma$$

$$= \int_0^T (F_x \delta\underline{x} + F_u \delta\underline{u}) \; dt + F\Delta T$$

Now from (1.23), the equation for $\underline{p}$ is

$$F_x = \dot{\underline{p}}' + \underline{p}'\underline{f}_x$$

Hence
$$J(\underline{x}_T + \Delta\underline{x}_T, \; T+\Delta T) - J(x_T, \; T)$$

$$= \int_0^T [(\dot{\underline{p}}' + \underline{p}'\underline{f}_x) \; \delta\underline{x} + F_u \delta\underline{u}] \; d\sigma + F\Delta T$$

$$= \underline{p}'(T) \; \delta\underline{x}(T) + \int_0^T \underline{p}'(\underline{f}_x \delta\underline{x} - \delta\dot{\underline{x}}) + F_u \delta\underline{u} \; d\sigma + F\Delta T$$

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From (1.9)

$$\underline{f}_x \delta \underline{x} - \delta \underline{\dot{x}} = - \underline{f}_u \delta \underline{u}$$

Hence

$$J(\underline{x}_T + \Delta \underline{x}_T, \; T+\Delta T) - J(\underline{x}_T, \; T) = p'(T) \; (\Delta \underline{x}_T - \underline{f} \Delta T)$$

$$+ \int_0^T (F_u - \underline{p}' \underline{f}_u) \; \delta \underline{u} \; d\sigma + F \Delta T$$

where use has been made of (1.34). But by (1.27), $F_u \; \underline{p}' \underline{f}_u = 0$; so,

$$J(x_T + \Delta x_T, \; T+\Delta T) - J(x_T, \; T) = \underline{p}' \Delta \underline{x}_T \; + (F - \underline{p}'(T)\underline{f}) \; \Delta T$$

$$= p'(T) \; \Delta \underline{x}_T - H \Delta T$$

$$= J_x \; \Delta \underline{x}_T + J_T \Delta T$$

where the last equality results from (1.40). This gives

$$J_x = p' \tag{1.43}$$

and

$$J_T = - H \tag{1.44}$$

In the space of variables $(x_T, \; T)$, the vector $\underline{p}'$ is the gradient of the function J; it is normal to the surfaces of constant J; H is the Hamiltonian of the function J. This sheds new light on the maximal principle. Along an optimal trajectory, the change in cost J over a given time step ΔT is a minimum, that is, H is a maximum.

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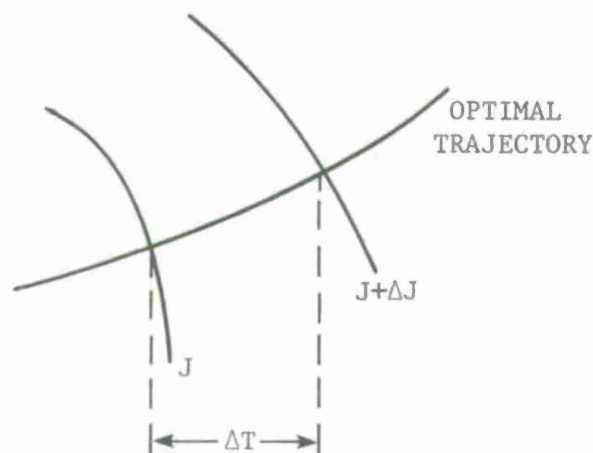
Figure 2 -- Constant Cost Fronts

These arguments hold only if the terminal cost is zero; G = 0.

RELATION TO DYNAMIC PROGRAMMING

The partial differential equations (1.43) and (1.44) can be obtained by the method of dynamic programming. This method is based on the Bellman principle of optimality.[7] According to the Bellman principle, an optimal control policy has the property that, regardless of the initial state or initial decision, the remaining decisions must constitute an optimal control policy with regard to the state which results from the first decision.

In terms of the cost function

$$C(\underline{u}) = \int_0^T F(\sigma, \underline{x}, \underline{u}) \, d\sigma$$

the Bellman principle takes the form.

The cost $C(\underline{u})$ is a minimum along a curve $\underline{x}$ defined on [0, T] if it is a minimum along each later part of the curve, that is, if

[7]Dreyfus, S. E., "Dynamic Programming and the Calculus of Variation," Academic Press, Inc., New York (1965).

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$$\int_{t}^{T} F(\sigma, \underline{x}, \underline{u}) \, d\sigma$$

is a minimum along the curve $\underline{x}$ on the interval $[t, T]$ for all $t\epsilon[0, T]$.

The integral is dependent on the end point $(t, x(t))$. If one defines

$$J(x, t) = \min_{\underline{u}} \int_{t}^{T} F(\sigma, \underline{x}, \underline{u}) \, d\sigma \qquad (2.1)$$

for all admisible controls u, then

$$J(x, t) = \min_{\underline{u}} \{F(t, x, \underline{u}) \, \delta t\} + \min_{\underline{u}} \int_{t+\delta t}^{T} F \, d\sigma$$

or

$$J(x, t) = \min_{\underline{u}} \{F(t, x, u) \, \delta t + J(x + \delta x, t + \delta t)\} \qquad (2.2)$$

This equation forms the basis of the direct methods for solving control problems, described by Dreyfus.[7] Larson[8] extended the direct methods to constrained problems.

If it is assumed that J has partial derivatives, the differential equations (1.43) and (1.44) can be obtained from (2.2). Hence, the boundary value problem for the optimal control is obtained. If the partial derivatives of J exist, the right-hand side of (2.2) can be expanded in a Taylor series:

$$J(x, t) = \min_{u} \{F\delta t + J(x, t) + J_{x}(x, t) \, \delta\underline{x} + J_{t}(x, t) \, \delta t\}$$

$$(2.3)$$

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[8]Larson, R. E., "State Increment Dynamic Programming," American Elsevier Publishing Company, Inc., New York (1968).

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From the differential equation (1.1)

$$\delta \underline{x} = \underline{f} \delta t$$

Hence

$$0 = \min_{\underline{u}} \{F + J_x \underline{f} + J_t\} \, \delta t$$

Since $\delta t > 0$,

$$0 = \min_{\underline{u}} \{F + J_x \underline{f} + J_t\} \qquad (2.4)$$

In order to find the minimum of the term in brackets, it is differentiated with respect to $\underline{u}$ and the result is set equal to zero. This is a necessary, but not sufficient condition; however, if one assumes a minimum, it serves the purpose.

$$F_u + J_x fu = 0 \qquad (2.5)$$

By (2.4),

$$F + J_x \underline{f} + J_t = 0 \qquad (2.6)$$

Hence

$$H_u = 0$$

$$\qquad (1.43)$$

$$J_t = - H = - (F + J_x f)$$

From (1.25)

$$J_x = p^T \qquad (1.42)$$

There is a difference between the definition of H here and its definition in the previous section. This is only an apparent difference in the sign of F, which occurs because the lower limit of the integral is used in the definition of J here rather than the upper limit as used earlier. Otherwise there is complete agreement with the results of the indirect method.

CONSTRAINTS ON THE CONTROL AND STATE VARIABLES

In most applications, the control or the state variables cannot be chosen arbitrarily but are subject to constraints. In the problem of a ship moving in a current, ship speed is limited by the maximum power available. The constraints can generally be expressed in terms of inequalities of the form

$$\underline{\phi}(\underline{x}, \underline{u}) \leq 0 \qquad (3.1)$$

where the vector inequality simply means that the components satisfy the inequality. The number of components in the vector $\underline{\phi}$ is the number of constraints on the system. The analysis does not depend on whether both $\underline{x}$ and $\underline{u}$ occur implicitly in the inequality; one can have constraints on the controls and not on the state of the system or vice versa without affecting the analysis.

In this presentation, the variables in the optimal control problem with constraints are the state variable $\underline{x}(t)$ and the control variable $\underline{u}(t)$ defined on an interval $0 \leq t \leq T$. The process being controlled is described by the dynamic equation (1.1):

$$\underline{\dot{x}}(t) = f(t, \underline{x}, \underline{u})$$

with initial condition $\underline{x} = \underline{x}_c$; the state and control variables are constrained by the inequality (3.1). For simplicity, the terminal cost is taken as zero, $G = 0$, and the cost function is given by the equation:

$$C_T(\underline{u}) = \int_0^T F(\sigma, \underline{x}, \underline{u}) \, d\sigma \qquad (3.2)$$

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The vector $\underline{f}$ and the cost function F are twice continuously differentiable with respect to $\underline{x}$ and continuously differentiable with respect to $\underline{u}$.

The Lagrange multipliers will be used here to reduce this problem to a two-point, boundary-value problem. As in (1.32), the differential equation is introduced into the cost function by means of a Lagrange multiplier $\underline{p}$.

$$C_p(\underline{u}) = \int_0^T F(\sigma, \underline{x}, \underline{u}) + \underline{p}'(\dot{\underline{x}} - \underline{f}(\sigma, \underline{x}, \underline{u}))\, d\sigma$$

which yields the variational equation

$$\int_0^T [F_x \delta_x + F_u \delta\underline{u} + \underline{p}'\, \delta\underline{x} - \underline{p}'\underline{f}_x \delta\underline{x} - \underline{p}'\underline{f}_u\, \delta\underline{u}]\, d\sigma$$

$$= \underline{p}'(T)\delta\underline{x}(T) + \int_0^T [(F_x - \dot{\underline{p}}' - \underline{p}'\underline{f}_x)\, \delta_x$$

$$+ (F_u - \underline{p}'\underline{f}_u)\, \delta\underline{u}\, d\sigma \geq 0 \tag{3.3}$$

The differential in the cost is greater than or equal to zero since it is assumed that the variation $\delta\underline{u}$ is around an optimal control, a control which minimizes the cost.

Because of the constraint (3.1), the vector $\delta\underline{u}$ is not free. For instance, suppose that for t between t_1 and t_2, the trajectory $\underline{z}(t)$ due to the optimal control $\underline{v}(t)$ is along the boundary of the allowable region; see Figure 3. One cannot freely choose the variation $\delta\underline{u}$ in the control vector for $t_1 \leq t \leq t_2$ and still expect to remain in the allowable region R.

For the optimal trajectory $\underline{z}$ and control $\underline{v}$, there are at most a finite number of intervals $t_k \leq t \leq t_k + 1$ such that equality holds for any of the equations in (3.1).* On such an interval, the conditions (3.1) can be split into two sets

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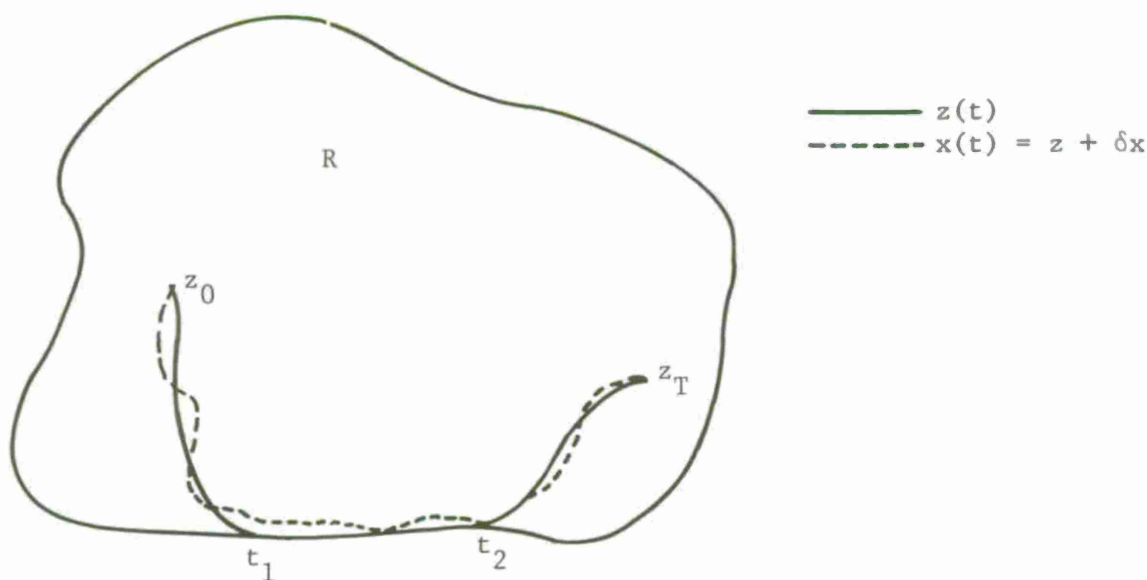*The proof of the statement is topological and beyond the scope of these notes.

Figure 3 -- Constrained Variables

$$\underline{\phi}_1(\underline{z}, \underline{v}) = 0$$

and

$$\underline{\phi}_2(\underline{z}, \underline{v}) < 0 \qquad\qquad (3.4)$$

where $\underline{\phi} = (\underline{\phi}_1, \underline{\phi}_2)$.

Consider a new vector $\underline{\psi}$ defined by

$$\underline{\phi}(\underline{x}, \underline{u}) + \underline{\psi}(\underline{x}, \underline{u}) = 0$$

The vector $\underline{\psi}$ is called a defect vector. Along the optimal trajectory, the vector $\underline{\psi}$ can also be split into two component vectors, $\underline{\psi}_1$ and $\underline{\psi}_2$, which correspond to the component vectors of $\underline{\phi}$. The component vectors of $\underline{\psi}$ also change from interval to interval. Along a given interval $[t_k, \ t_{k+1}]$

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$$\underline{\psi}_1 = 0$$

$$\underline{\psi}_2 > 0 \tag{3.5}$$

Since $\underline{\phi}_1(\underline{z}, \underline{v})$ is zero on this interval, either $\underline{z}$, $\underline{v}$, or both are on the boundary of their allowable range. From previous arguments, it is known that one cannot freely choose $\delta\underline{u}$. Only those values of $\delta\underline{u}$ are allowed which satisfy

$$\underline{\phi}_1(\underline{z} + \delta\underline{x}, \ \underline{v} + \delta\underline{u}) \leq 0$$

or by (3.4)

$$\underline{\phi}_1(\underline{z} + \delta\underline{x}, \ \underline{v} + \delta\underline{u}) - \underline{\phi}_1(\underline{z}, \ \underline{v}) \leq 0$$

On the other hand, for a neighboring trajectory to $\underline{z}$

$$\underline{\phi}(\underline{z} + \delta\underline{x}, \ \underline{v} + \delta\underline{u} \ + \underline{\psi} + \delta\underline{\psi} = 0$$

on $[t_k, \ t_{k+1}]$. Since $\underline{\phi} = -\underline{\psi}$

$$\phi(\underline{z} + \delta\underline{x}, \ \underline{v} + \delta\underline{u}) - \phi(\underline{z}, \ \underline{v}) + \delta\underline{\psi} = 0 \tag{3.6}$$

In order that the above inequality and (3.6) hold,

$$\delta\underline{\psi}_1 \geq 0 \tag{3.7}$$

Moreover, provided the variations are sufficiently small, $\delta\underline{\psi}_2$ is free.

If $\underline{\phi}$ is twice continuously differentiable, then it follows from (3.6) that

$$\underline{\phi}_x \ \delta x + \underline{\phi}_u \ \delta\underline{u} + \delta\underline{\psi} = 0 \tag{3.8}$$

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Set $\delta \underline{u} = (\delta \underline{u}_1, \delta \underline{u}_2)$ and consider the first N_{ϕ_1} equations in (3.8), $N_{\phi_1} = \dim (\underline{\phi}_1)$.

$$\underline{\phi}_{1x} \, \delta \underline{x} + \underline{\phi}_{1u_1} \, \delta \underline{u}_1 + \underline{\phi}_{1u_2} \, \delta \underline{u}_2 + \delta \underline{\psi}_1 = 0$$

If the square matrix ϕ_{1u_1} is not singular, its inverse γ exists, and

$$\delta \underline{u}_1 = - \gamma \, \underline{\phi}_{1u_2} \, \delta \underline{u}_2 - \gamma \, \underline{\phi}_{1x} \, \delta \underline{x} - \gamma \, \delta \underline{\psi}_1$$

The vectors $\delta \underline{u}_2$ and $\delta \underline{x}$ are free; the vector $\delta \underline{\psi}_1$ satisfies (3.7). If the matrix $\underline{\phi}_{1u_1}$ is singular, the first N_{ϕ_1} constraints were dependent; eliminate the dependent constraints and start again.

The contribution to the cost differential (3.3) from the interval $t_k \leq t \leq t_{k+1}$ is the following integral:

$$I_k = \int_{t_k}^{t_{k+1}} \{[F_x - \dot{\underline{p}}' - \underline{p}'\underline{f}_x - F_{u_1} \gamma \, \underline{\phi}_{1x} - \underline{p}'\underline{f}_{u_1} \gamma \, \underline{\phi}_{1x}] \, \delta \underline{x}$$

$$+ [(F_{u_2} - \underline{p}'\underline{f}_{u_2} - F_{u_1} - \underline{p}'\underline{f}_{u_1}) \gamma \, \underline{\phi}_{1u_2}] \, \delta \underline{u}_2$$

$$- (F_{u_1} - \underline{p}'\underline{f}_{u_1}) \gamma \, \delta \underline{\psi}_1\} \, d\sigma$$

Define the vector $\underline{\lambda}_1$ by

$$\underline{\lambda}_1' = - (F_{u_1} - \underline{p}'\underline{f}_{u_1}) \gamma \tag{3.10}$$

Then

$$I_k = \int_{t_k}^{t_{k+1}} \{[F_x - \dot{\underline{p}}' - \underline{p}'\underline{f}_x + \underline{\lambda}_1'\underline{\phi}_{1x}] \, \delta_x$$

$$+ [(F_{u_2} - \underline{p}'\underline{f}_{u_2}) + \underline{\lambda}_1' \, \underline{\phi}_{1u_2}] \, \delta u_2 + \underline{\lambda}_1'\delta \underline{\psi}_1\} \, d\sigma$$

The vector $\underline{p}$ can be determined so that the coefficient of $\delta\underline{x}$ vanishes:

$$\underline{\dot{p}}' = \underline{F}_x - \underline{p}' \ \underline{f}_x + \underline{\lambda}_1' \underline{\phi}_{1x} \tag{3.11}$$

Since $\delta\underline{u}_2$ is free, the usual argument that $\delta\underline{u}_2$ is zero everywhere except on a small interval yields

$$F_{u_2} - \underline{p}' \ \underline{f}_{u_2} + \underline{\lambda}_1' \ \underline{\phi}_{1u_2} = 0 \tag{3.12}$$

Now $\delta\underline{u}$ can be chosen so that $\delta\underline{u} = 0$ for $t \leq t_k$ and for $t_{k+1} \leq t$. In this case, the only contribution to the cost difference (3.3) is that due to I_k; hence

$$0 \leq I_k = \int_{t_k}^{t_{k+1}} \underline{\lambda}_1' \ \underline{\delta\psi}_1 \ d\sigma$$

By (3.7), $\underline{\delta\psi}_1 \geq 0$; so

$$\underline{\lambda}_1 \geq 0 \tag{3.13}$$

Let the Hamiltonian be defined by

$$H = -F + \underline{p}' \ \underline{f} - \underline{\lambda}' \ \underline{\phi} \tag{3.14}$$

where $\underline{\lambda}$ is defined by

$$\lambda_j \geq 0 \quad \text{if} \ \phi_j = 0$$

$$\lambda_j = 0 \quad \text{if} \ \phi_j < 0$$

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The differential system (1.28) also holds for this H, that is,

$$\dot{\underline{x}} = \frac{\partial H}{\partial p}$$

$$\dot{\underline{p}} = -\frac{\partial H}{\partial x} \qquad (1.28)$$

$$0 = \frac{\partial H}{\partial u}$$

One example of a constrained control problem is that of a forced harmonic oscillator in which the magnitude of the force is limited. In this problem, the force is the control and the process is one of changing the velocity and displacement of the harmonic oscillator. It becomes an optimal control problem if one is interested in finding the force or control which reduces the oscillator from a given velocity and displacement to zero velocity and displacement in minimum time.

The equation of motion for the forced harmonic oscillator with a limited force is simply

$$m \frac{d^2 z}{dt^2} + cz = F$$

where $|F| \leq M$, a given constant. Set $x = cz/M$, $\tau = \omega t$, and $u = F/M$ where $\omega = \sqrt{c/M}$. In terms of these nondimensional variables, the nondimensional form of the equation of motion is

$$\ddot{x} + x = u \qquad (3.15)$$

where the control function satisfies the inequality $|u| \leq 1$. This constraint can also be written in the form

$$\phi_1(u) = (u - 1) \leq 0$$

$$\phi_2(u) = -(u + 1) \leq 0 \tag{3.16}$$

The optimal control problem can be formulated in the phase plane. If (x, y) are the phase plane coordinates, the equation of motion (3.15) takes the form

$$\dot{x} = y$$

$$\dot{y} = -x + u \tag{3.17}$$

Starting the oscillator at a given displacement with a given velocity is equivalent to assigning a given point $(x, y) = (a, b)$ in the phase plane as an initial condition for (3.17). The rest state of the oscillator is represented in the phase plane by the point $(0, 0)$, the point of zero displacement and velocity. Hence, the optimal time control problem is one of finding a control u which minimizes the time between states (a, b) and $(0, 0)$. In this problem, the cost is given by

$$C_T(u) = T = \int_0^T d\tau \tag{3.18}$$

The cost function $F(\tau, x, u) = 1$.

Set $\underline{p} = (p, q)$. Then the Hamiltonian defined by (3.14) is

$$H = -1 + py + q(u - x) - \lambda(u - 1)(u + 1) \tag{3.19}$$

and, moreover, (1.28) takes the form

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$$\dot{x} = \frac{\partial H}{\partial p} = y$$

$$\dot{y} = \frac{\partial H}{\partial q} = -x + u$$

$$\dot{p} = -\frac{\partial H}{\partial x} = q$$

$$\dot{q} = -\frac{\partial H}{\partial y} = -p$$

$$0 = \frac{\partial H}{\partial u} = q - \lambda(u - 1) - \lambda(u + 1) \tag{3.20}$$

Suppose u is an optimal control which reduces the oscillator from the state (a, b) to the state $(0, 0)$ in the minimal time T, and suppose $|u| < 1$ for the interval $T_0 \leq \tau \leq T_1$. Suppose $q \neq 0$ on $T_1 \leq \tau \leq T_2$. By (3.20), $q - 2\lambda u = 0$; hence, $\lambda \neq 0$ on (T_1, T_2). A consequence of $\lambda \neq 0$ is that $\phi = 0$; hence, if $q \neq 0$, it follows that $|u(t)| = 1$ on (T_1, T_2). In other words, one needs to look only for the optimal control among those controls for which $|u(t)| = 1$.

Now $|u| = 1$ implies $u = \pm 1$; hence, the solution of (3.20) is given as:

$$x \mp 1 = A \sin (\tau + \alpha)$$

$$y = A \cos (\tau + \alpha)$$

$$p = B \sin (\tau + \alpha_0)$$

$$q = B \cos (\tau + \alpha_0)$$

$$q = 2 \lambda u \tag{3.21}$$

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Since $\lambda \geq 0$, it follows from the last of these equations that the sign of q is the same as the sign of u. Hence, if q changes from positive to negative, the optimal control must switch from +1 to -1. It switches from -1 to +1 if q changes from negative to positive.

In a neighborhood of the origin, the optimal trajectory satisfies

$$(x \mp 1)^2 + y^2 = 1$$

Hence, its final segment is either on the circle of radius 1 about $(-1, 0)$, or it is on the circle of radius 1 about $(1, 0)$; see Figure 4. Suppose for the sake of argument that there is an $\varepsilon > 0$ such that $u(\tau) = -1$ for $T - \varepsilon \leq \tau < T$. The last segment of the optimal trajector is on the semicircle $\{(x + 1)^2 + y^2 = 1, 0 \leq y\}$..

Between $(0, 0)$ and $(-2, 0)$, the parameter τ would change along this semicircle by the amount π; hence, the sign of q must change somewhere on this semicircle. At the point S_1 where q changes sign, the sign of u must also change, and u switches from -1 to 1. The optimal path continues backward on the circle of radius r_1 around $(1, 0)$ until either (a, b) is reached or q changes sign. But q does not change sign until the point S_2 is reached since the time between S_1 and S_2 is π. At S_2, the control would switch to -1 and the optimal trajectory would continue back on the circle of radius r_2 around $(-1, 0)$. This process is continued until the point (a, b) is reached. In the process, one switches control each time one of the following semicircles is intercepted:

$$[x - (2n - 1)]^2 + y^2 = 1, \quad y \geq 0, \quad n=0,1,2,\ldots \quad (3.22)$$

or

$$[x + (2n - 1)]^2 + y^2 = 1, \quad y \leq 0 \quad n=0,1,2,\ldots \quad (3.23)$$

The curve formed by these semicircles is called the switching curve; see Figure 5.
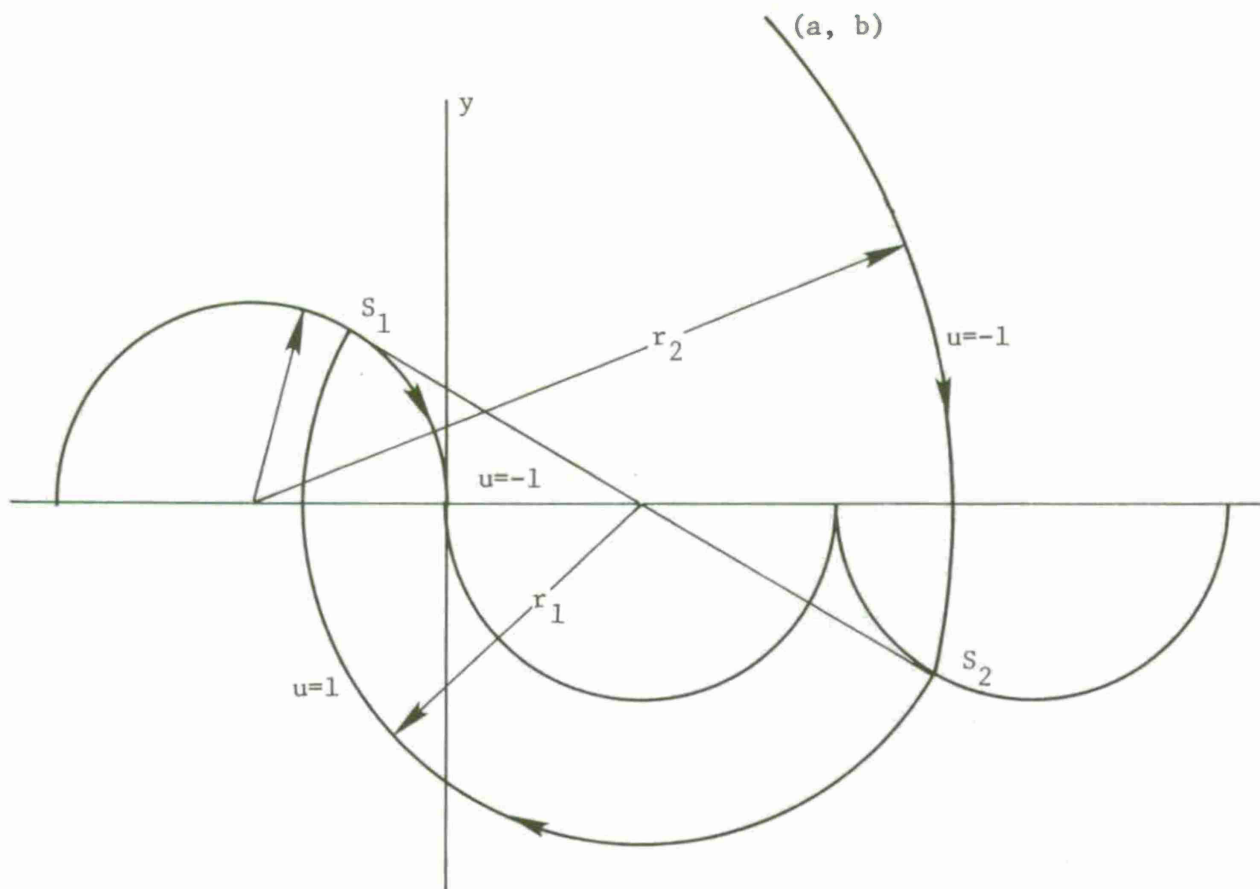
Figure 4 -- Optimal Trajectory
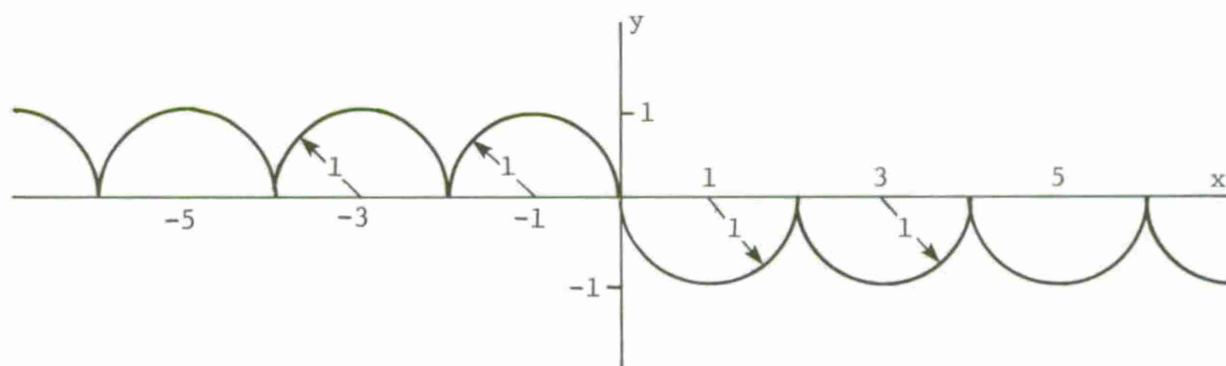
Figure 5 -- Switching Curve

The optimal control and the resulting trajectory in the phase plane can now be obtained by reversing the above procedure. If (a, b) is above the switching curve, proceed with the control u = - 1. The optimal trajectory will be along the circle

$$(x + 1)^2 + y^2 = (a + 1)^2 + b^2$$

in the direction of that part of the switching curve which lies to the right of x = 0. For (a, b) on the switching curve, use u = - 1 if x < 0 or u = 1 if x > 0. If (a, b) lies below the switching curve, start with u = 1 and change to u = - 1 at the switching curve. Change the sign of u at each intersection with the switching curve. When u = 1, the optimal trajectory lies on a circle with center at (1, 0); when u = - 1, it is on a circle around (-1, 0).

Suppose only one switch in u is needed to reach the origin from (a, b). Because of the symmetry of the problem geometry in the phase plane, it is necessary to consider only those cases for which a = 1 after the switch. The origin is then approached along the trajectory

$$x = 1 - \cos (T - \tau)$$

$$y = - \sin (T - \tau) \tag{3.24}$$

which is on the semicircle $\{(x, y) \mid (x - 1)^2 + y^2 = 1, \ y \leq 0\}$ let τ_s be the time at which the switch occurs. The optimal trajectory for $\tau \leq \tau_s$ is given by

$$x = - 1 + A \sin (\tau + \alpha)$$

$$y = A \cos (\tau + \alpha) \tag{3.25}$$

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where A and α are constants defined by

$$A \sin \alpha = a + 1$$

$$A \cos \alpha = b$$

By (3.24) and (3.25), the switching time must satisfy

$$1 - \cos (T - \tau_s) = 1 + A \sin (\tau_s + \alpha)$$

$$- \sin (T - \tau_s) = A \cos (\tau_s + \alpha)$$

Elimination of τ_s from these equations yields a relationship between the terminal time T and the initial point (a, b), namely,

$$(a + 1 + \cos T)^2 + (b + \sin T)^2 = 4 \qquad (3.26)$$

By definition, time fronts are the curves which connect initial points having the same terminal time T. Equation (3.26) can be used to determine the time fronts for $T \leq \pi$. If T = 0, the time front is simply the origin; if there are no switches in the control, the initial point is an endpoint of the curve connecting all initial points from which the origin is reached with one switch in time T. More than one switch would require $T > \pi$. From (3.26), the time fronts for $0 < T \leq \pi$ are segments of the circle of radius 2 around the point (-1 - cos T, - sin T); see Figure 6. It is the segment of the circle which lies above the switching path. At the switching path, the time front is tangent to the vertical line x = constant for x > 0; at the opposite end, it is tangent to the switching curve. For $T = \pi$, the time front is a circle of radius 2 around the origin.
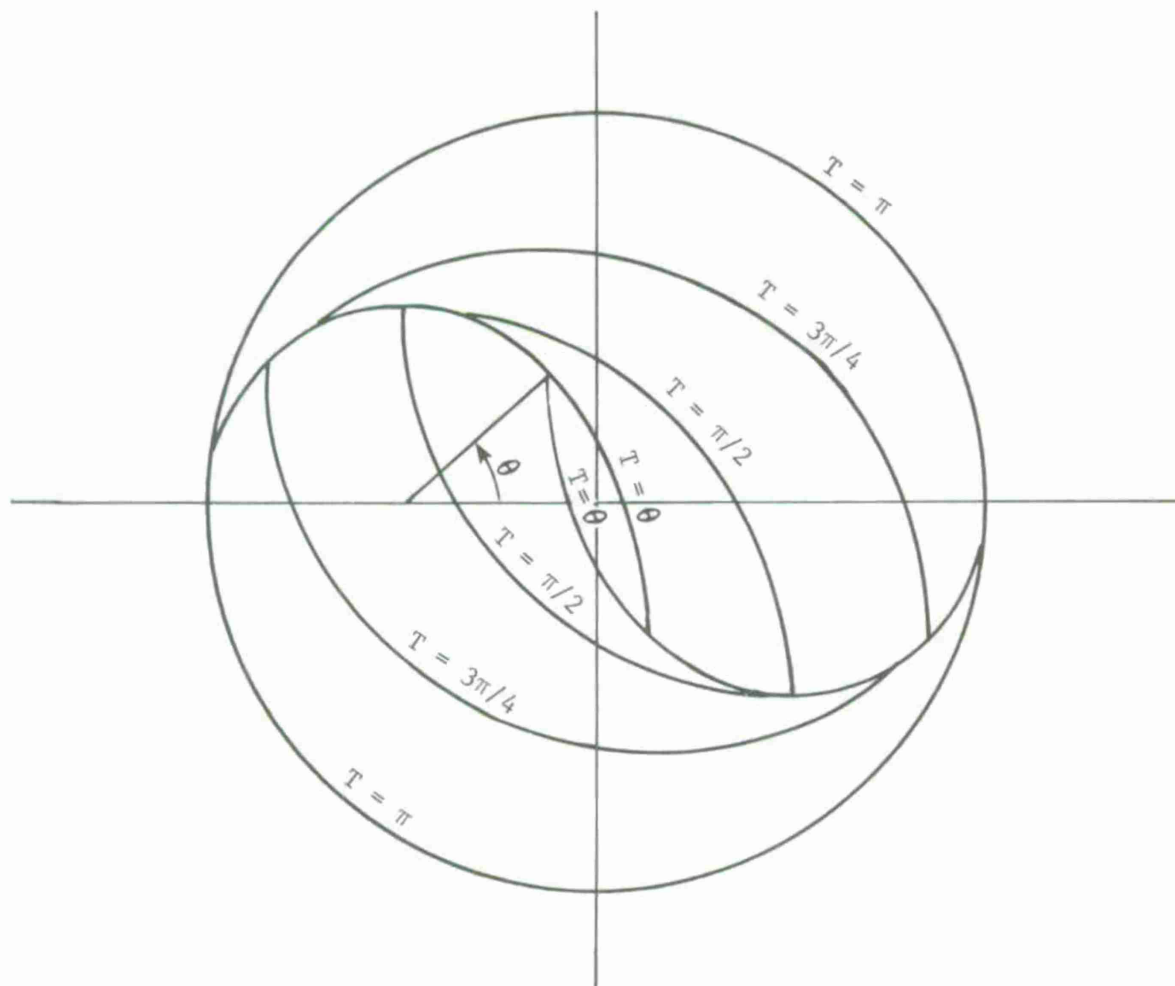
Figure 6 -- Time Fronts

STOCHASTIC SYSTEMS

Stochastic control theory was first applied in this country at the Massachusetts Institute of Technology during World War II to synthesize fire control systems. In the 1960's it was applied to space navigation, guidance, and orbit determination in such well-known missions as Ranger, Mariner, and Apollo. Applications of the filtering theory, aspects of control theory include submarine navigation, fire control, aircraft navigation, practical schemes for detection theory, and numerical integration. There have also been industrial applications; one example involved the problem of basic weight control in the manufacture of paper.[1]

The filtering and prediction theory developed by Wiener and Kolmogorov forms the cornerstone of stochastic control theory. It provides an estimate of the signal or the state of a process on the basis of observation of the signal additively corrupted by noise. Unfortunately, the Wiener-Kolmogorov theory cannot be applied extensively because it requires the solution of the Wiener-Hopf integral equation. It is difficult to obtain closed form solutions to this equation, and it is not an easy equation to solve numerically.

Kalman and Bucy[2] give a solution to the filtering problem under weaker assumptions than those of the original Wiener problem. Their solution makes it possible to solve prediction and filtering problems recursively and is ideally suited for digital computers. Basically, it can be viewed as an algorithm which, given the observation process, sequentially computes in real time the conditional distribution of the signal process. The estimated state of the process is given as the output of a linear dynamical system driven by the observations. One determines the coefficients for the dynamical system by solving an initial value problem for a differential equation. This differential equation is easier to solve than the Wiener-Hopf equation.

Our attention here will be limited to linear systems with quadratic cost functions. In this case the solution of the optimal control

problem is given by the separation theorem.[1] The solution consists of
an optimal filter for estimating the state of the system from the ob-
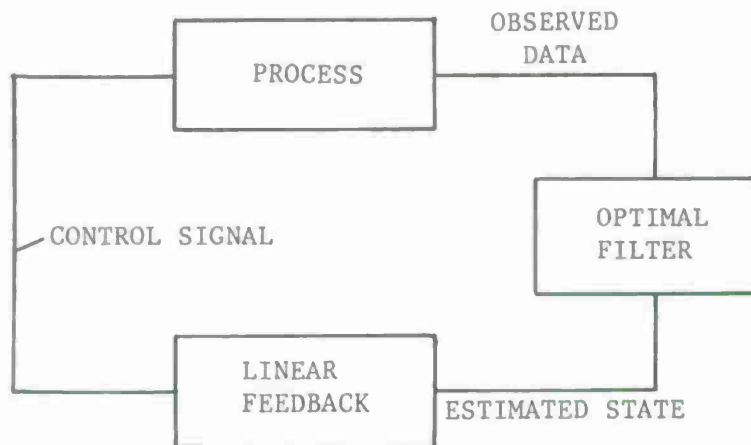served data and a linear feedback of the estimated state of the system;
see Figure 7.

Figure 7 -- Stochastic Control System

The optimal filter is the Kalman-Bucy filter, which will be dis-
cussed in detail in the next section; the linear feedback is the same as
would be obtained if the state of the system could be measured exactly
and if there were no random disturbances in the system. Thus, the
linear feedback can be determined by solving a deterministic problem.
Because of time limitations, we will not prove but merely accept the
separation theorem.

One objection to the use of stochastic control theory is that the
process to which the theory is applied may not be random but merely
irregular. For instance, the traffic flow on the Washington Beltway may
not be truely random but it is certainly highly irregular. If I need to
reach Dullis Airport from DTNSRDC by 1 pm, it might take me 45 to 50
minutes; but to reach the airport at 6 pm, I would have to allow 2
hours. The reason for this variation in lead time is that there will be
bumper-to-bumper traffic on the Beltway during the rush hour and any
accident brings this traffic to a halt. It is not the microscopic but
the macroscopic properties of the traffic flow that govern our lead time
estimate. The traffic flow could be analyzed as a stochastic process;
such a model would be acceptable provided it predicted the macroscopic

properties of the flow. This is analogus to using linear models in the deterministic case. If the predictions agree with the experimental results, the linear theory is said to be good; if they do not, then the process is said to be nonlinear. In using a statistical model, one should recognize that it is only a model and not the actual process, and one should continually strive to determine the accuracy of his models.

There are many reasons in favor of applying stochastic theory. The solution of the stochastic problem may be possible whereas the deterministic theory may be hopelessly impossible. In many problems such as that of traffic flow, one may not be interested in the microscopic properties but merely in certain macroscopic properties. In the control problem, the stochastic model distinguishes between open and closed looped systems but the deterministic model does not. Another reason for using a stochastic model may be that this model is closer to the physics of the actual situation.

In any case the purpose of this section is to lay the ground work for stochastic control theory. Our attention will be focused on certain concepts of stochastic processes and random differential equations.

To describe a stochastic process rigorously would require measure theory and a great deal more time. Our approach will therefore not be rigorous, but hopefully it will be complete enough to get across the basic ideas. For the rigorous approach, see either Doob[9] or Gikhman and Skorokhod.[10]

A real random variable ξ is a set of numbers or events together with a probability measure defined on this set. It is characterized by its distribution function $F(x)$ which is defined by

$$F(x) = P \{\xi \leq x\}$$

[9]Doob, J. L., "Stochastic Processes," Wiley, Inc., New York (1963).

[10]Gikhman, I. I. and A. V. Skorokhod, "Introduction to the Theory of Random Processes," W. B. Saunders Company, Philadelphia, Pa. (1969).

where $P\{\xi \leq x\}$ is the probability that ξ is less than or equal to x. The distribution function is nonnegative, nondecreasing, and continuous from the left; also $F(-\infty) = 0$ and $F(\infty) = 1$.

Analogously, if ξ is an n-truple of random variables, its distribution function is a function of n real variables.

$$F(x_1, x_2, \ldots, x_n) = P\{\xi_1 \leq x_1, \ldots, \xi_n \leq x_n\}$$

and F is called a joint distribution function of the variables ξ_k. The function $F(x_1, x_2, \ldots, x_n)$ is uniquely defined in n-dimensional Euclidian space E_n, is non-decreasing, and is continuous from the left with respect to each variable. Furthermore,

$$F(x_1, x_2, \ldots x_i, -\infty, x_{i+2}, \ldots x_n) = 0$$

and

$$F(x_1, \ldots, x_i, \infty, \ldots, \infty) = F^{(i)}(x_1, \ldots, x_i)$$

where $F^{(i)}$ denotes the distribution function of the i-truple $(\xi_1, \ldots, \xi_i)$.

A random function or a stochastic process is a random variable $\xi(t)$ which is a function of time. As time varies, $\xi(t)$ describes the evolution of the process. If a random process is recorded as it evolves, the recorded function $\xi(\cdot)$ describes only one of the many possible ways in which the process might have developed. The recorded function $\xi(\cdot)$ is called a sample function of the random process. For each fixed value of t, the quantity $\xi(t)$ is a random variable.

Whereas a random variable is characterized by a distribution function, a stochastic process is characterized by a set of joint distribution functions. Assume that it is possible to assign a probability distribution to the multidimensional random variable $\xi^n = (\xi(t_1), \xi(t_2), \ldots, \xi(t_n))$ for any n and arbitrary times t_i. The distribution function

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$$F(x_1, x_2, \ldots, x_n; t_1, \ldots, t_n) = P \{\xi(t_1) \leq x_1, \ldots, \xi(t_n) \leq x_n\}$$

is called the finite-dimensional distribution of the stochastic process $\xi(t)$. For F to be a distribution, it must satisfy the following compatibility conditions:

$$F(x_1, x_2, \ldots, x_i, \infty, \ldots, \infty; t_1, \ldots, t_n) = F(x_1, x_2, \ldots, x_i; t_1, \ldots, t_n)$$

for $i \leq n$ and

$$F(x_1, \ldots, x_n; t_1, \ldots, t_n) = F(x_{j_1}, \ldots, x_{j_n}; t_{j_1}, \ldots, t_{jn})$$

where $j_1, \ldots, j_n$ is an arbitrary permutation of the indicies 1, 2, ..., n. The mean value of a stochastic process is defined by

$$m(t) = E[\xi(t)] = \int_{-\infty}^{\infty} x \, d \, F(x, t)$$

where E is the mathematical expected value. The mean value is thus a function of time. Higher moments of ξ are defined similarly.

The covariance of the stochastic process is given by

$$r(s, t) = \text{cov} [\xi(t), \xi(s)] = E [(\xi(t) - m(t)) (\xi(s) - m(s))]$$

$$= \iint_{-\infty}^{\infty} (x - m(t)) (y - m(s)) \, d \, F(x, y; t, s)$$

Our definition of a stochastic process is very general, and most systems which come under this definition would be mathematically unmanageable. Some specialization of the theory which makes it possible to characterize the distribution of $\xi(t_1), \xi(t_2), \ldots, \xi(t_n)$ in a simple way are particularly attractive. For instance, if the distribution of

$\xi(t_1),\ldots,\xi(t_n)$ is identical to the distribution of $\xi(t_1 + \tau)$, $\xi(t_2 + \tau),\ldots,\xi(t_n + \tau)$ for all τ and all arbitrary choices of the times $t_1,\ldots,t_n$, then the stochastic process $\xi(t)$ is said to be stationary. If only the first and second moments $E[\xi]$ and $E[\xi^2]$ of the distributions are equal, then the process is weakly stationary.

Our discussion of control systems has been limited to systems in which knowledge of the system at time t together with the governing equations suffices to describe its future evolution. Knowledge of the past when the present is given is superfluous relative to the future evolution of the system. The stochastic system analogy of this situation is the Markov property for random processes; these are stochastic process-es in which the past and future of the processes are conditionally independent. In order to define a Markov process, the conditional probability and the transition probabilities have to be defined. The conditional probability $P(A|B)$ is the probability that A will occur if B has occurred. Given a sequence of times $t_1 < t_2 < \ldots < t_n < t$, the probability that $\xi(t) \leq x$ if the sample function $\xi(\cdot)$ has already taken the values $\xi(t_1)$, $\xi(t_2),\ldots,\xi(t_n)$ is denoted by $P(\xi(t) \leq x|\xi(t_1),\ldots, \xi(t_n))$. A stochastic process is said to be a Markov process if

$$P(\xi(t) \leq x|\xi(t_1),\ldots, \xi(t_n)) = P(\xi(t) \leq x|\xi(t_n))$$

The transition probability distribution $F(x, t|y, s)$ is defined by $F(x, t|y, s) = P(\xi(t) \leq x|\xi(s) = y)$. If a stochastic process is a Markov process, its finite distribution functions are given by

$$F(x_1, x_2,\ldots, x_n; t_1,\ldots, t_n) =$$
$$F(x_1; t_1) \, F(x_2, t_2|x_1, t_1)\ldots F(x_n, t_n|x_{n-1}, t_{n-1})$$

This results from an application of the Baye rule. A Markov process is thus defined by two functions, the absolute probability distribution $F(x, t)$ and the transition probabilities $F(x, t | y, s)$.

Consider a system with the following dynamic equation:

$$\underline{\dot{x}} = \underline{f}(t, \underline{x}, \underline{u}) + \xi \underline{w} (t) \qquad (4.1)$$

where ξ is a small parameter and w is a stochastic process. Since w is stochastic, the state of the system x will also be stochastic; thus, we are interested in solving stochastic differential equations. Furthermore, our interest is not with a particular sample function $x(\cdot)$ which is a particular discription of the state of the system during one run through the process; our interest is with the statistical properties of the stochastic process $x(t)$.

Consider the linear stochastic differential equation

$$dx = A x \, dt + dw \qquad (4.2)$$

where w is a stochastic process. In order to make some progress in finding the statistical properties of x, assume that w is a Wiener process.

A Wiener process is a Markov process which satisfies the following conditions:

1. It is a second order process; that is, for all t

$$E[w^2(t)] < \infty$$

Hence, the mean $m(t)$ exists as well as the covariance function

$$r(s, t) = \text{cov} [w(t), w(s)]$$

2.   The process has independent increments; that is, for arbitrary
times $t_1 < t_2 < \ldots < t_n$, the increments

$$x(t_n) - x(t_{n-1}), \; x(t_{n-1}) - x(t_{n-2}), \ldots, x(t_2) - x(t_1), \; x(t_1)$$

are independent.*

3.   The distribution of $x(t) - x(s)$ for arbitrary t and s depends
only on t−s.   In this case, the process is said to have stationary
increments.

4.   The transition probabilities are Gaussian.   In the one-
dimensional case, the transition probability density is

$$p(t + \Delta t, \; w \mid t, \; 0) = \frac{1}{\sqrt{2\pi \Delta t}} \; \exp - w^2/2\Delta t$$

5. $w(0) = 0$ with probability one, and $E[w(t)] = 0$ for all $t > 0$.

Sample functions of a Wiener process have interesting properties.
They can be continuous functions but are nowhere differentiable.   Their
paths are of infinite length.   Yet it is for just such perturbations
that (4.2) will be solved.

If w in (4.2) had bounded variation, the solution could be written
in terms of the transport matrix $\Phi(x, \; t)$ of the linear system

$$\dot{y} = A \; y \qquad\qquad (4.3)$$

The solution of (4.2) would be

$$\underline{x}(t) = \Phi(t, \; 0) \; \underline{c} + \int_0^t \Phi(t, \; \tau) \; d \; w(\tau) \qquad (4.4)$$

where the value of x at t = 0 is the random variable c.   The expectation
of c is m and its covariance matrix is Γ.

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*Independent random variables are defined on page 7 of Doob.[9]

The integral

$$\int_0^t \Phi(t, \tau) \, d \, w(\tau)$$

is a stochastic integral. Since the transport matrix $\Phi(t, \tau)$ is deterministic and has continuous derivatives, one way of defining this integral is through integration by parts.

$$\int_0^t \Phi(t, \tau) \, d \, w(\tau) = \Phi(t, t) \, w(t) - \Phi(t, 0) \, w(0)$$

$$- \int_0^t \frac{\partial \Phi}{\partial t} (t, \tau) \, w(t) \, d\tau$$

It follows from (1.15) and other properties of the transport matrix that

$$\int_0^t \Phi(t, \tau) \, d \, w(\tau) = w(t) - \Phi(t, 0) \, w(0) + \int_0^t \Phi(t, \tau) \, A(\tau) \, w(\tau) \, d\tau$$

$$(4.5)$$

The integral on the right exist for almost all sample functions since the sample functions of $w(t)$ are almost all continuous. This way of defining the integral has the desirable feature that the integral can be interpreted as an integral of sample functions. It does not, however, preserve the intuitive idea that the integral is a limit of sums of independent random variables nor can it be extended to the case where Φ is stochastic. Doob gives a more formal definition of the integral together with detailed proofs of its stochastic properties.

The expected value of this integral is computed as follows:

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$$E\left[\int_0^t \Phi(t, \tau) \, d \, w(\tau)\right] = E[w(t)] - \Phi(t, 0) \, E[w(0)]$$

$$+ E\left[\int_0^t \Phi(t, \tau) \, A(\tau) \, w(\tau) \, dT\right]$$

$$= m(t) - \Phi(t, 0) \, m(0) + \int_0^t \Phi(t, \tau) \, A(\tau) \, m(\tau) \, d\tau$$

Hence

$$E\left[\int_0^t \Phi(t, \tau) \, d \, w(\tau)\right] = \int_0^t \Phi(t, \tau) \, d \, m(\tau) \qquad (4.6)$$

The properties of the solution of the stochastic differential equation (4.4) will now be investigated. Since x is a linear function of a normal process, it is also normal and can be characterized completely by the mean value function and the covariance function. Since the expected value of the Wiener process w(t) is zero,

$$E[x(t)] = \Phi(t, 0) \, E[c] + E\left[\int_0^t \Phi(t, T) \, d \, w(\tau)\right]$$

$$= \Phi(t, 0) \, m_0$$

where m_0 is the expected value of the initial condition c. Hence

$$m_x(t) = E[x(t)] = \Phi(t, 0) \, m_0 \qquad (4.7)$$

Taking derivatives yields

$$\frac{dm_x}{dt} = \frac{d}{dt} \, \Phi(t, 0) \, m_0 = A(t) \, \Phi(t, 0) \, m_0 = A(t) \, m_x \qquad (4.8)$$

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Thus the mean value satisfies the linear differential Equation (4.3).

The covariance matrix is more difficult to compute. In order to simplify the calculations, assume $m_0 = 0$; hence, $E[x(t)] = 0$. This can always be achieved by subtracting m_x from x. For $s \geq t$,

$$R(s, t) = cov\ [\underline{x}(s),\ \underline{x}(t)] = E[\underline{x}(s)\ \underline{x}^T(t)]$$

$$= E\left[\left\{\Phi(s,\ t)\ \underline{x}(t)\ +\ \int_t^s\ \Phi(s,\ \sigma)\ d\ w(\sigma)\right\}\ \underline{x}^T(t)\right]$$

$$= \Phi(s,\ t)\ E[x(t)\ x^T(t)]\ +\ \int_t^s\ \Phi(s,\ \sigma)\ E[d\ w(\sigma)\ \underline{x}^T(t)]$$

$$= \Phi(s,\ t)\ R(t,\ t) \tag{4.9}$$

The integral is zero since $\underline{w}(\sigma)$ and $\underline{x}(t)$ are independent for $s \geq t$. Set $P(t) = R(t,\ t) = E[\underline{x}(t)\ \underline{x}^T(t)]$. Then $P(t)$ is the variance and is therefore the function of interest.

$$P(t) = E\left[\left(\Phi(t,\ 0)\ c\ +\ \int_0^t\ \Phi(t,\ \tau)\ d\ w(\tau)\right)\right.$$

$$+\ \left.\left(\Phi(t,\ 0)\ c\ +\ \int_0^t\ \Phi(t,\ \sigma)\ d\ w(\sigma)\right)^T\right]$$

$$= \Phi(t,\ 0)\ E[c\ c^T]\ \Phi^T(t,\ 0)$$

$$+\ \Phi(t,\ 0)\ E\left[c\ \int_0^t\ d\ w^T(\sigma)\ \Phi^T(t,\ \sigma)\right.$$

$$+\ \int_0^t\ \Phi(t,\ \tau)\ E[d\ w(\tau)\ c^T]\ \Phi^T(t,\ 0)$$

$$+\ \int_0^t\ \int_0^\tau\ \Phi(t,\ \tau)\ E[d\ w(\tau)\ d\ w^T(\sigma)]\ \Phi^T(t,\ \sigma)$$

The increments of the Wiener process are independent of C; hence

$$E[c\ d^T\ w(\sigma)] = E[d\ w(\tau)\ c^T] = 0$$

Moreover, from the properties of the Wiener process

$$E[d\ w(T)\ d\ w^T(\sigma)] = 0$$

if $d\tau$ and $d\sigma$ have no parts in common; otherwise

$$E[d\ w(T)\ d\ w^T(\tau)] = R_w\ d\tau$$

where R_w is the covariance matrix of the Wiener process w. The final expression for P is then

$$P(t) = \Phi(t,\ 0)\ \Gamma\ \Phi^T(t,\ 0) + \int_0^t \Phi(t,\ \tau)\ R_w(\tau)\ \Phi^T(t,\ \tau)\ d\tau \tag{4.10}$$

A differential equation for P can be obtained from this expression for P simply by differentiating

$$\frac{dP}{dt} = \left[\frac{d}{dt}\ \Phi(t,\ 0)\right]\ \Gamma\ \Phi^T(t,\ 0) + \Phi(t,\ 0)\ \Gamma\ \frac{d}{dt}\ \Phi^T(t,\ 0)$$

$$+ \Phi(t,\ t)\ R_w(t)\ \Phi^T(t,\ t) + \int_0^t \frac{\partial\ \Phi(t,\ \tau)}{\partial t}\ R_w(\tau)\ \Phi^T(t,\ \tau)\ dT$$

$$+ \int_0^t \Phi(r,\ \tau)\ R_w(\tau)\ \frac{d}{dt}\ \Phi^T(t,\ \tau)\ d\tau$$

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The transport matrix satisfies

$$\frac{\partial \Phi}{\partial t} (t, \tau) = A \, \Phi(t, \tau)$$

and

$$\frac{\partial \, \Phi^T(t, \tau)}{\partial t} = \Phi^T(t, \tau) \, A^T$$

Hence

$$\frac{dP}{dt} = A \, \Phi(t, 0) \, \Gamma \, \Phi^T(t, 0) + \Phi(t, 0) \, \Gamma \, \Phi^T(t, 0) \, A^T$$

$$+ \, R_w(t) + \int_0^t A \, \Phi(t, T) \, R_w(\tau) \, \Phi^T(t, \tau) \, d\tau$$

$$+ \int_0^t \Phi(t, \tau) \, R_w(\tau) \, \Phi^T(t, \tau) \, A^T \, d\tau$$

$$\frac{dP}{dt} = A \left\{ \Phi(t, 0) \, \Gamma \, \Phi^T(t, 0) + \int_0^t \Phi(t, \tau) \, R_w(T) \, \Phi^T(t, \tau) \, d\tau \right\}$$

$$+ \left\{ \underline{\Phi}(t, 0) \, \Gamma \, \Phi^T(t, 0) + \int_0^t \Phi(t, \tau) \, R_w \Phi^T(t, \tau) \, dT \right\} A^T$$

$$+ \, R_w(t)$$

Thus from (4.10)

$$\frac{dP}{dt} = A \, P + P \, A^T + R_w \tag{4.11}$$

$$P(0) = \Gamma \tag{4.12}$$

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THE KALMAN-BUCY FILTER

The solution of the optimal control problem for a linear stochastic
system is given by the separation theorem. It consists of an optimal
filter for estimating the state of the system from the observed data and
a linear feedback of the estimated state of the system; see Figure 7.
The linear feedback is the same as the feedback that would be obtained
if there were no stochastic perturbation of the system. This section
will develop the explicit computational schemes for solving the filter-
ing problem.

Suppose we have the stochastic process described in the previous
section

$$dx = A \ x \ dt + d \ w(t) \tag{5.1}$$

$$x(0) = c \tag{5.2}$$

where $w(t)$ is a Wiener process and c is a Gaussion zero mean n-vector.
In an actual case in which the process is realized, it is important to
know the state of the system. It is, however, not always possible to
measure x directly; instead, a set of quantities $z(t)$ dependent on x are
measured. Assume that the dependence of z on x is linear and is given
by

$$dz = H \ x \ dt + dv \tag{5.3}$$

where the perturbation v is a Wiener process independent of x.

The filter problem can be formulated as follows. Assume that a
realization of the output z has been observed over the interval
$0 \leq \tau \leq t$. Determine the best estimate of the value of the state vector
x at time t. It is assumed here that the admissible estimates of x are
linear functionals $F(z)$ of the observed output z. The criterion

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for determining the best estimate is that the mean square estimation error be a minimum. This best estimate $\hat{x}(t)$ is dependent on the values of $z(\tau)$ in the interval $0 \leq \tau \leq t$, and it can be proved that it is a linear combination of the values of z on this interval.

$$\hat{x}(t) = \int_0^t K(t, \tau) \, d \, z(\tau) \qquad (5.4)$$

Since $z(\tau)$ is a stochastic variable, $\hat{x}(t)$ is a stochastic integral.

Interpolation and extrapolation are two problems that are related to the filtering problem. The interpolation problem is one of estimating the state at some time $\tau < t$; the extrapolation problem is one of estimating it at some time $\tau > t$. This latter problem is the one which is of interest to the stock market investor.

The condition that $\hat{x}(t)$ is the best estimate from among all linear functionals of $z(t)$ for the state vector x in the least squares sense is stated mathematically as follows. For every constant vector λ and linear functional F,

$$E[\{\lambda^T(x(t) - \hat{x}(t))\}^2] \leq E[\{\lambda^T(x(t) - F(z))\}^2] \qquad (5.5)$$

where all variables have a zero mean.

$$E[\hat{x}(t)] = E[x(t)] = E[F(z)] = 0$$

Now set

$$\tilde{x} = x - \hat{x}$$

where $\tilde{x}$ is called the minimum error vector.

$$E[(\lambda^T \tilde{x})^2] \leq E[\lambda^T(\tilde{x} + (F(z) - \hat{x}))^2]$$

$$\leq E[(\lambda^T \tilde{x})^2] + 2E[\lambda^T \tilde{x} \, \lambda^T (F(z) - \hat{x})]$$

$$+ E[(\lambda^T (F(z) - \hat{x}))^2]$$

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For all λ and $F(z)$, the criterion (5.5) requires

$$E[(\lambda^T (F(z) - \hat{x}))^2] + 2E[\lambda^T \tilde{x} \lambda^T (F(z) - \hat{x})] \geq 0$$

This can be true only if

$$0 \equiv E[\lambda^T \tilde{x} \lambda^T (F(z) - \hat{x})] = \lambda^T E[\tilde{x}(F(z) - \hat{x})^T]\lambda$$

But this implies that

$$E[\tilde{x} (F(z) - \hat{x})^T] = 0$$

for any linear combination $F(z)$ of elements of z; hence

$$E[\tilde{x} F^T(z)] = 0 \qquad (5.6)$$

An integral equation for the kernel $K(t, \tau)$ can be derived from (5.6). This kernel is not a stochastic quantity, and it can be determined independent of the realization $z(\cdot)$. For $F(z) = z(\tau) - z(\sigma)$, $0 \leq \sigma \leq \tau \leq t$, the expression (5.6) yields

$$E[x(t) (z(\tau) - z(\sigma))^T] = E[\hat{x}(t) (z(\tau) - z(\sigma))^T]$$

$$= E\left[\hat{x}(t) \left\{ \int_\sigma^\tau H(s) x(s) ds + d v(s) \right\}^T \right]$$

$$= E\left[\left\{ \int_0^t K(t, r) dz(r) \right\} \left\{ \int_\sigma^\tau H(s) x(s) ds + dv(s) \right\}^T \right]$$

$$= E\left[\int_0^t \int_\sigma^\tau K(t, r) (H(r) x(r) dr + dv(r)) (H(s) x(s) ds + dv(s))^T \right]$$

$$= E\left[\int_0^t \int_\sigma^\tau K(t, r) H(r) x(r) x^T(s) H^T(s) ds dr \right.$$

$$+ \int_0^t \int_\sigma^\tau K(t, \ r) \ H(r) \ x(r) \ dv^T(s) \ dr$$

$$+ \int_0^t \int_\sigma^\tau K(t, \ r) \ dv(r) \ x^T(s) \ H^T(s) \ ds$$

$$+ \int_0^t \int_\sigma^\tau K(t, \ r) \ dv(r) \ dv^T(s) \Bigg]$$

From the properties of Wiener processes,

$$E \int_0^t \int_\sigma^\tau K(t, \ r) \ dv(r) \ dv^T(s) = \int_\sigma^\tau K(t, \ s) \ R_v(s) \ ds$$

where R_v is the covariance matrix of the process. Furthermore, $dv(s)$ and $x(s)$ are independent, so $E \ dv(r) \ x^T(s) = 0$; hence

$$E[x(t) \ (z(\tau) - z(\sigma))^T] = \int_\sigma^\tau \Bigg\{ \int_0^t K(t, \ r) \ H(r) \ E[x(r) \ x^T(s)]$$

$$(5.7)$$

$$H^T(s) \ dr + K(t, \ s) \ R_v(s) \Bigg\} \ ds$$

for all σ and τ. On the other hand, from (5.3)

$$E[x(t) \ (z(\tau) - z(\sigma))^T] = E \Bigg[x(t) \Bigg\{ \int_\sigma^\tau H(s) \ x(s) \ ds + dv(s) \Bigg\}^T \Bigg]$$

$$= \int_\sigma^\tau E[x(t) \ x^T(s)] \ H^T(s) \ ds$$

$$= \int_\sigma^\tau \Bigg\{ \int_0^t K(t, \ r) \ H(r) \ E[x(r) \ x^T(s)] \ H^T(s) \ dr$$

$$+ K(t, \ s) \ R_v(s) \Bigg\} \ ds$$

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where the last equality results from (5.7). Since this equation holds for all σ and τ in the interval [0, t],

$$K(t, s) \, R_v(s) = E[x(t) \, x^T(s)] \, H^T(s) - \int_0^t K(t, r) \, H(r) \, E[x(r) \, x^T(s)] \, H^T(s) \, dr$$

(5.8)

This is a nonhomogeneous integral equation for K(t, s). Its kernel is $H(r) \, E[x(r) \, x^T(s)] \, H^T(s)$. Since it corresponds to a positive definite quadratic form, all its eigenvalues are positive and the equation has a solution. Unfortunately, it is not possible to calculate K(t, s) from this equation because $E[x(r) \, x^T(s)]$, the covariance of x(s), is unknown.

A different equation for K(t, s) can be obtained from (5.8) by differentiating both sides of it with respect to t.

$$\frac{\partial K(t, s)}{\partial t} \, R_v(s) = \frac{\partial}{\partial t} E[x(t) \, x^T(s)] \, H^T(s)$$

$$- K(t, t) \, H(t) \, E[x(t) \, x^T(s)] \, H^T(s)$$

$$- \int_0^t \frac{\partial K(t, r)}{\partial t} \, H(r) \, E[x(r) \, x^T(s)] \, H^T(s) \, dr$$

By (5.1)

$$dx = Ax \, dt + dw$$

Hence

$$E[dx(t) \, x^T(s)] \, H^T(s) = A(t) \, E[x(t) \, x^T(s)] \, H^T(s) \, dt$$

$$+ E[dw(t) \, x^T(s)]$$

The second term vanishes since $dw(t)$ and $x^T(s)$ are independent if $t \geq s$. This yields

$$\frac{\partial K(t, s)}{\partial t} R_v(s) = [A(t) - K(t, t) H(t)] E[x(t) x^T(s)] H^T(s)$$

$$- \int_0^t \frac{\partial K}{\partial t} (t, r) H(r) E[x(r) x^T(s)] H^T(s) dr$$

Use of the integral equation (5.8) to obtain an expression for $E[x(t) x^T(s)] H^T(s)$ yields

$$\left[\frac{\partial K(t, s)}{\partial t} + K(t, t) H(t) K(t, s) - A(t) K(t, s) \right] R_v(s)$$

$$= - \int_0^t \left[\frac{\partial K(t, r)}{\partial t} + K(t, t) H(t) K(t, r) - A(t) K(t, r) \right]$$

$$H(r) E[x(r) x^T(s)] H^T(s) dr \qquad (5.9)$$

Set

$$\overline{\psi}(t, s) = \frac{\partial K(t, s)}{\partial t} + K(t, t) H(t) K(t, s) - A(t) K(t, s)$$

Then by (5.9)

$$\overline{\psi}(t, s) = - \int_0^t \overline{\psi}(t, r) H(r) E[x(r) x^T(s)] H^T(s) R_v^{-1}(s) dr$$

$$(5.10)$$

Since the kernel of this integral equation corresponds to a positive definite quadratic form, the only solution of (5.10) is

$$\overline{\psi}(t, s) \equiv 0$$

This yields the following differential equation for $K(t, s)$:

$$\frac{\partial K(t, s)}{\partial t} = A(t) K(t, s) - K(t, t) H(t) K(t, s) \qquad (5.11)$$

From the integral equation (5.8) for $K(t, s)$

$$K(t, t) R_v(t) = E[x(t) x^T(t)] H^T(t)$$

$$- \int_0^t K(t, r) H(r) E[x(r) x^T(t)] H^T(t) dr$$

On the other hand,

$$E[\hat{x}(t) x^T(t)] = E\left[\int_0^t K(t, r) dz(r) x^T(t)\right]$$

$$= \int_0^t K(t, r) H(r) E[x(r) x^T(t)] dr$$

$$+ \int_0^t K(t, r) E[dv(r) x^T(t)]$$

where the second integral vanishes; hence

$$K(t, t) R_v(t) = E[x(t) x^T(t)] H^T(t) - E[\hat{x}(t) x^T(t)] H^T(t)$$

$$= E[(x(t) - \hat{x}(t)) x^T(t)] H^T(t)$$

$$= \{E[\tilde{x}(t) \hat{x}^T(t)] + E[\tilde{x}(t) \tilde{x}^T(t)]\} H^T(t)$$

The condition that $\hat{x}$ is the best estimate from among all linear func-
tionals of $z(t)$ leads to the result that $E[\tilde{x}(t)\ \hat{x}^T(t)] = 0$. Hence

$$P(t)\ H^T(t) = E[\tilde{x}(t)\ \tilde{x}^T(t)]\ H^T(t) = K(t,\ t)\ R_v(t) \quad (5.12)$$

From the stochastic integral,

$$\hat{x}(t) = \int_0^t K(t,\ r)\ dz(r)$$

$$d\hat{x}(t) = K(t,\ t)\ dz(t) + \int_0^t \frac{\partial\ K(t,\ r)}{\partial t}\ dz(r)\ dt$$

$$= P\ H^T\ R_v^{-1}\ dz(t) + \int_0^t (A(t)\ K(t,\ r) - K(t,\ t)\ H(t)\ K(t,\ r))\ dz(r)\ dt$$

or

$$d\hat{x}(t) = A(t)\ \hat{x}(t)\ dt + P\ H^T\ R_v^{-1}(dz(t) - H(t)\ \hat{x}(t)\ dt) \quad (5.13)$$

Since $z(t)$ and presumably $dz(t)$ are known, this is a stochastic differ-
ential equation for $\hat{x}(t)$.

Note that

$$dz(t) - H(t)\ \hat{x}(t)\ dt = dz(t) - H(t)\ x(t)\ dt + H(t)\ \tilde{x}(t)\ dt$$

$$= dv(t) + H(t)\ \tilde{x}(t)\ dt$$

From this expression and (5.13), we get the following stochastic
differential equation for $\tilde{x}$

$$d\tilde{x} = dx(t) - d\hat{x}(t)$$

$$= A \, x \, dt + dw - A \, \hat{x} \, dt - P \, H^T R_v^{-1} \, (dv + H(t) \, \tilde{x})$$

$$= A \, \tilde{x} \, dt + dw - P \, H^T \, R_v^{-1} \, dv - P \, H^T R_v^{-1} \, H \, \tilde{x} \, dt$$

$$= [A - P \, H^T \, R_v^{-1} \, H] \, \tilde{x} \, dt + dw - P \, H^T \, R_v^{-1} \, dv$$

with $\tilde{x}(0) = x(0)$. By the methods developed in the previous section for stochastic differential equations,

$$P(t) = E[\tilde{x}(t) \, \tilde{x}^T(t)]$$

$$= \Phi(t, \, 0) \, \Gamma \, \Phi^T(t, \, 0) + \int_0^t \Phi(t, \, \sigma) \, [Q(\sigma)$$

$$+ P(\sigma) \, H^T(\sigma) \, R_v^{-1}(\sigma) \, H(\sigma) \, P^T] \, \Phi^T(t, \, \sigma) \, d\sigma \qquad (5.14)$$

where $\Phi(t, \, \tau)$ is the transport matrix associated with the linear differential equation

$$\frac{dy}{dt} = (A - P \, H^T \, R_v^{-1} \, H) \, y \qquad (5.15)$$

and where

$$Q \, d\tau = E[dw(\tau) \, dw^T(\tau)] \qquad (5.16)$$

from (4.11)

$$\frac{dP}{dt} = (A - P \, H^T \, R_v^{-1} \, H) \, P + P(A - P \, H^T \, R_v^{-1} \, H)^T$$

$$+ Q(t) + P \, H^T \, R_v^{-1} \, H \, P^T$$

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or

$$\frac{dP}{dt} = A\ P + P\ A^T - P\ H^T\ R_v^{-1}\ H\ P - Q \qquad (5.17)$$

$$P(0) = \Gamma \qquad (5.18)$$

This set of equations finishes the solution of the filter problem.

The optimal filter is a feedback system which is described by the stochastic differential equation (5.13). It is obtained by taking the measurements $z(t)$, forming the error signal $z(t) - H(t)\ \hat{x}(t)$, and feeding the error forward with a gain $P(t)\ H^T(t)\ R_v^{-1}(t)$. $P(t)$, the error variance, is obtained as a solution to the nonlinear Riccati-type equation (5.17), $H(t)$ is a known transformation matrix, and R_v is the variance of the Wiener process dv. A block diagram of the filter is shown in Figure 8. The variables appearing in this diagram are vectors, and the boxes represent matrices operating on vectors. The double lines which indicate direction of signal flow serve as a reminder that multiple signals rather than a single one are being directed.
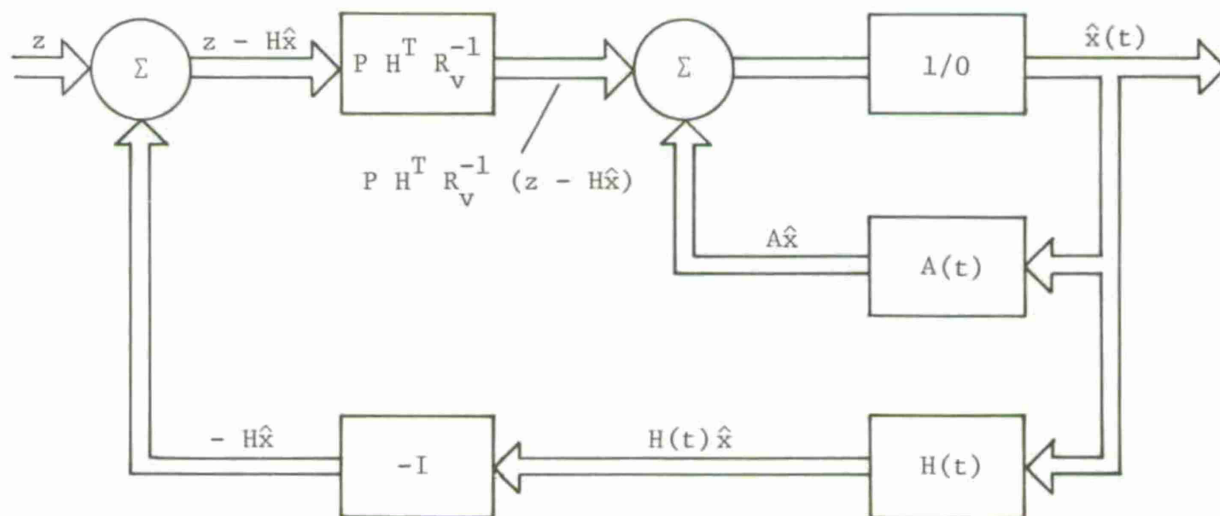
Figure 8 -- Optimal Filter

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